AAPT UNITED STATES PHYSICS TEAM
AIP 2022

36^{th} Annual
US Physics Team Training Camp
College Park, Maryland

THEORETICAL EXAMINATION

- This is the main theoretical team selection test for the 2022 US Physics Traveling Team.
- The time limit is 5 hours. There are 3 problems, which are each worth an equal amount of points.
- Before you start the exam, make sure you are provided with blank paper, both for your answers and scratch work, writing utensils, a hand-held scientific calculator with memory and programs erased, and a computer for you to download the exam.
- At the end of the exam, you have 20 minutes to upload solutions to all of the problems. For each problem, scan or photograph each page of your solution, combine them into a single PDF file, and upload them.

Reference table of possibly useful information

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_{\text{m}} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_{\text{B}} = 1.38 \times 10^{-23}$ J/K
$N_{\text{A}} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_{\text{A}} k_{\text{B}} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \theta^3/6$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \theta^2/2$ for $\lvert\theta\rvert \ll 1$

You may use this sheet throughout the exam.

Take Five

The five parts of this question are unrelated.

1. Consider a partially polarized light beam, containing a mix of unpolarized and linearly polarized light. The intensity of the beam is analyzed using a linear polarizer. At a particular orientation of the polarizer, the outgoing beam has maximum intensity I_{max}. Turning the polarizer by a 30° angle reduces the outgoing beam's intensity by 10%. Find the degree of polarization of the beam,

$$V \equiv \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}$$

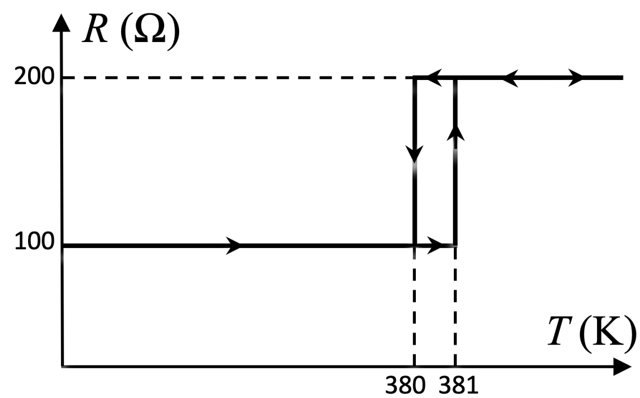
where I_{min} is the minimum intensity for any orientation of the polarizer.

2. According to Newton's law of cooling, a hot object transfers heat to the environment at a rate

$$P = k(T_o - T_l),$$

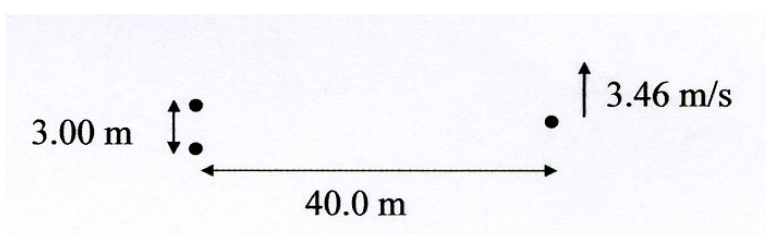
where T_o is the temperature of the object, T_l is the temperature of the environment, and k is a constant.

Consider a circuit element whose resistance R depends on its temperature T as shown below (not to scale), with heat capacity $C = 2$ J/K in a lab of temperature $T_l = 270$ K. Note that the curve is multivalued, which indicates hysteresis: the resistance takes the lower value when increasing from low temperatures, and the higher value when decreasing from high temperatures.

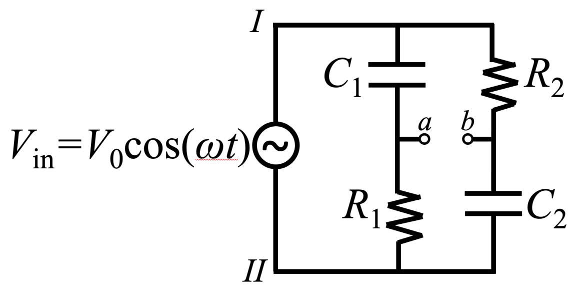


When this component is placed in series with a voltage $V_1 = 50$ V, its temperature stabilizes at $T_1 = 350$ K. When it is instead placed in series with a voltage $V_2 = 70$ V, its temperature does not stabilize, and the current through it instead oscillates. Find the period of these oscillations.

3. Two speakers are 3.00 m apart. They both emit perfect sinusoids, whose frequencies differ by 0.250 Hz. Spaceman Fred, who is standing 40.0 m away in the direction shown in the diagram, must run at 3.46 m/s to avoid hearing beats. The speed of sound in air is 343 m/s. Approximately what frequency are the speakers emitting?

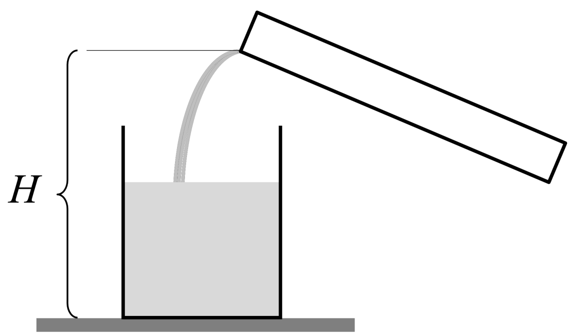


4. The input voltage is the voltage difference between points I and II. What is the voltage difference between the points a and b in the circuit below as a function of time?



You should assume $R_1C_1 = R_2C_2$, and simplify your answer as much as possible.

5. An empty cylindrical glass of cross-sectional area A is resting on a table. Water of density ρ is slowly poured into the glass from a beaker, at a constant volume per unit time Q.



The beaker nozzle is at height H, and the water exits the beaker with negligible speed. Let $t = 0$ at the moment the water first hits the bottom of the glass. Find the force of the water on the glass as a function of time, until the glass overflows. Assume the water does not splash and the atmospheric pressure is P_0. Furthermore, assume the glass is wide, so that the rate at which the water level rises is negligible compared to the speed at which water enters the glass.

Chain Reaction

The three parts of this question are unrelated.

1. A frictionless circular cylinder of radius R is placed with its axis horizontal, and a flexible, inextensible string of uniform linear mass density λ is wrapped around it. When the length of the string is slightly longer than $2\pi R$, part of the string will sag below the cylinder. Now suppose the string is slowly shortened, until the entire string just touches the cylinder. At this moment, find the tension at the top of the string.

2. Model a grappling hook as a point mass m attached to the end of a uniform chain of linear mass density λ. Initially, the chain is loosely coiled on the ground. Then the mass is launched directly upward from the ground, with an initial speed v_0. The chain is flexible, so that when the mass is at a height y, a length y of the chain dangles directly beneath it, while the rest of the chain remains at rest on the ground. Find the maximum height reached by the mass, assuming this is less than the length of the chain. (Hint: if you directly compute the acceleration, you will find an intractable differential equation, but it can be solved with a clever change of variable.)

3. A uniform rod of length $2R$ is placed inside a fixed, frictionless hemispherical bowl of radius R. In equilibrium, the rod makes an angle θ with the horizontal. Assume that the rod and bowl are ideally rigid, but that the lip of the bowl and the end of the rod are both slightly rounded, so that there is a well-defined normal direction at the points they touch. Find an analytic expression for θ and evaluate it to three significant figures, in degrees.

Lego Movie

Gravitational waves are predicted by general relativity, but can be modeled with Newtonian physics and a few small assumptions. Throughout this problem, assume classical Newtonian physics, and ignore special relativistic effects. The mass of the sun is $M_\odot = 2.0 \times 10^{30}$ kg, and the luminosity of the sun is $L_\odot = 3.8 \times 10^{26}$ W. In all parts of this problem, you may use the fundamental constants c and G in your answers.

1. Consider a spherically symmetric body with mass M. Determine the Schwarzschild radius R_s of such a body so that the escape velocity would be equal to the speed of light c.

2. Two such bodies, with masses M_1 and M_2, are in circular orbits about their common center of mass. The separation of the bodies is R, and the total mass is $M = M_1 + M_2$.

   (a) Find the frequency f of the orbital motion in terms of M and R.

   (b) Find the total energy E of the system in terms of M_1, M_2, and R.

   (c) The minimum possible orbital separation is $R_{\text{min}} = R_1 + R_2$, where R_1 and R_2 are the Schwarzschild radii for masses 1 and 2. Find the maximum possible orbital frequency f_{max} in terms of M.

3. We would like to estimate the rate at which the system loses energy due to the emission of gravitational waves. In classical electromagnetism, the simplest form of radiation is dipole radiation, which results from a second time derivative of the electric dipole moment. However, for gravity the analogue of the electric dipole moment is the center of mass, which always moves at constant velocity by momentum conservation. Thus, the leading source of gravitational radiation is quadrupole radiation, which depends on a time derivative of the moment of inertia. All subparts of this part are rough estimates, which means you may drop numeric prefactors such as π.

   (a) The power radiated in gravitational waves by a system with moment of inertia I takes the form[1]

   $$P = kG^\alpha c^\beta \left(\frac{d^n I}{dt^n}\right)^2$$

   where k is a dimensionless constant. Determine α, β, and n.

   (b) For two black holes circularly orbiting each other in the xy plane, with center of mass at the origin, find the moment of inertia $I_y(t)$ about the y-axis in terms of M_1, M_2, R, and the angular frequency ω, defining the origin of time so that $I_y(0) = 0$.

   (c) Roughly estimate the average power radiated over an orbital period. Your final answer should be a product of powers of M_1, M_2, M, R, and fundamental constants.

   (d) The maximum power radiated occurs when $R = R_{\text{min}}$. Roughly estimate the maximum power in the case $M_1 = M_2$ in terms of fundamental constants. What is the order of magnitude of its ratio to the luminosity of the Sun?

4. The energy loss due to gravitational wave emission causes orbiting black holes to spiral towards each other, changing the orbital frequency over time. Assume the orbit is always approximately circular.

[1] Technically, the exact answer does not contain the moment of inertia, but a more complex object called the reduced quadrupole moment. However, the two are close enough for the rough estimates in this problem.

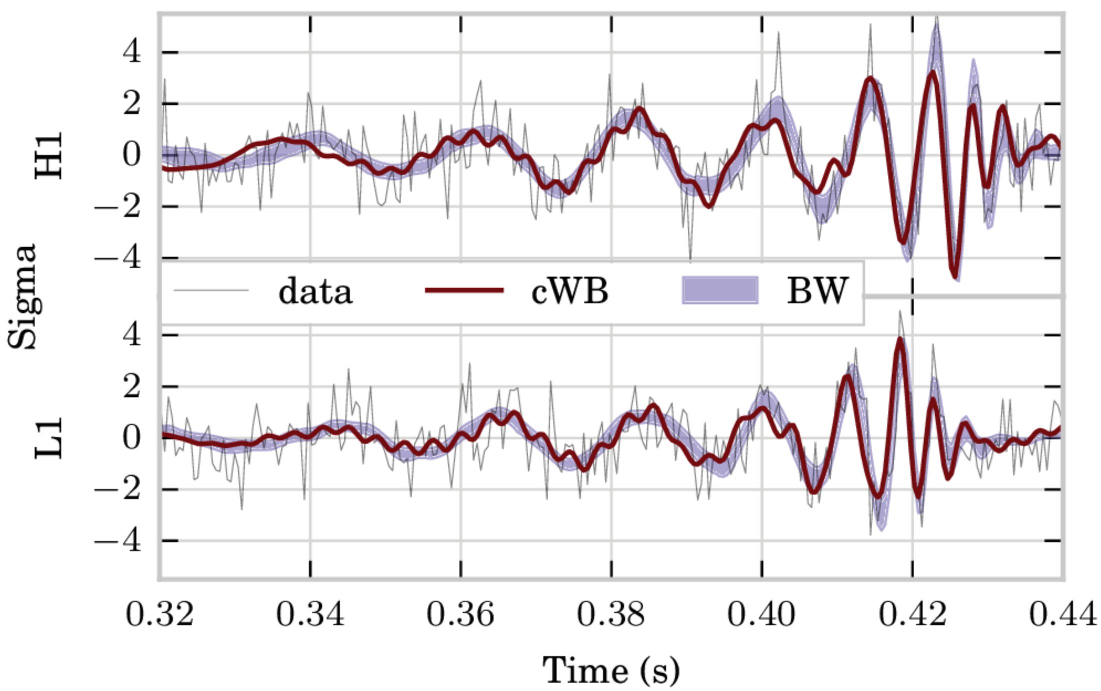
(a) Assuming the energy loss is slow, find the rate of change of the orbital frequency df/dt in terms of f, M_1 and M_2. You should find your answer is simply expressed in terms of the "chirp mass" M_c, defined as

$$M_c = \left(\frac{M_1^3 M_2^3}{M_1 + M_2}\right)^{1/5}.$$

Here you are expected to keep numeric prefactors. In particular, according to general relativity, the correct numeric prefactor to part 3(c) is 32/5.

(b) What is the frequency f_g of the gravitational waves emitted when the orbital frequency is f?

(c) The Hanford, Washington and Livingston, Louisiana LIGO detectors observed a binary black hole merger event on September 14, 2015. Their data is shown in the graphs marked H1 and L1. Use the smoothed (shaded) H1 data to answer the questions below. No detailed data analysis is expected.



Graph downloaded from LIGO Open Science Center, operated by California Institute of Technology and Massachussets Institute of Technology and supported by the U. S. National Science Foundation: losc.ligo.org

i. Estimate the maximum gravitational wave frequency, and thereby estimate the total mass M, giving your answer as a multiple of the solar mass $M_\odot$.

ii. Estimate the chirp mass M_c, giving your answer as a multiple of the solar mass $M_\odot$.