AAPT UNITED STATES PHYSICS TEAM
AIP 2009

Quarterfinal Exam

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Important Instructions for the Exam Supervisor

- This examination consists of one part.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet during the exam.
- Allow 60 minutes to complete the exam. Examinees may read the cover sheet before beginning the exam, but may not look at the examination questions until the 60 minute time period begins.
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after March 8, 2009.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.
- **The examinees will need to use a ruler for one of the questions on this exam.** They may *not* share rulers with other examinees.
- AAPT *must* receive the students answer papers no later than Thursday, March 5, 2009. Marking of papers will occur that weekend, and the semifinalists will be selected by March 8, 2009. It will not be possible to mark late papers.

AAPT UNITED STATES PHYSICS TEAM
AIP 2009

2009 Quarter-Final Exam

4 QUESTIONS - 60 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

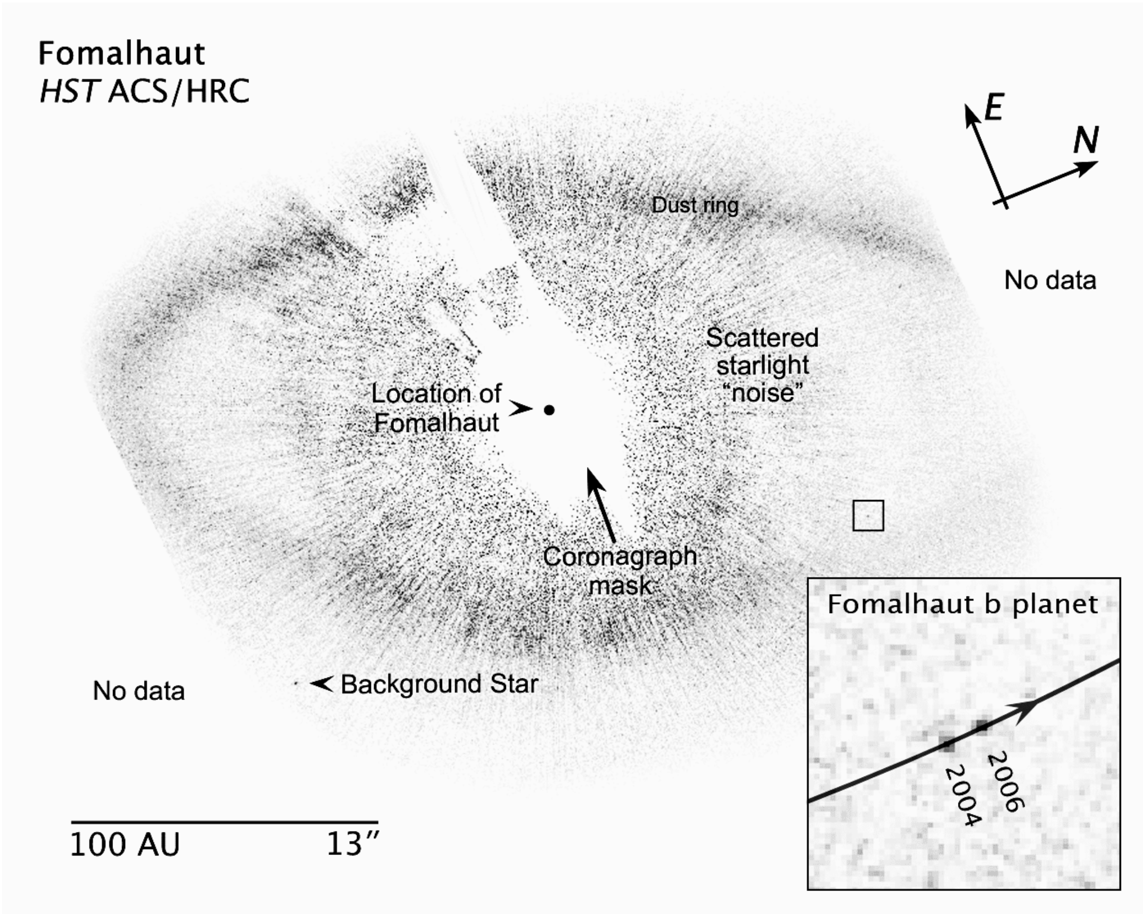
- Show all your work. Partial credit will be given.
- Start each question on a new sheet of paper. Put your name in the upper right-hand corner of each page, along with the question number and the page number/total pages for this problem. For example,

  Doe, Jamie

  Prob. 1 - P. 1/3

- A ruler will be required on this exam.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared.
- Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Each of the four questions is worth 25 points. The questions are not necessarily of the same difficulty. Good luck!
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after March 10, 2009.**

1. Below is an image of Fomalhaut b, the first extrasolar planet to be observed directly by visible light, obtained by the Hubble Space Telescope.



Astronomy Picture of the Day 2008 November 14 http://antwrp.gsfc.nasa.gov/apod/ap081114.html
Image Credit: NASA, ESA, P. Kalas, J. Graham, E. Chiang, E. Kite (Univ. California, Berkeley), M. Clampin (NASA/Goddard), M. Fitzgerald (Lawrence Livermore NL), K. Stapelfeldt, J. Krist (NASA/JPL)

The scale of the larger diagram is shown on the lower left; the 13" refers to the angle, in arc-second, subtended by 100 AU at the distance of Fomalhaut. The scale of the inset can be determined by size of the small box.

(a) A planet is in a circular orbit of radius R and period T. Derive an expression for the mass of a star in terms of R, T, and the gravitational constant G. You may assume that the mass of the planet is much, much less than the mass of the star.

(b) From the image, estimate the mass of the star Fomalhaut in solar masses. You may assume that the orbit of Fomalhaut b is circular and ignore any errors associated with the fact that the plane of the image is not coincident with the plane of the orbit. Recall that the radius of the Earth's orbit around the Sun is 1 AU. You do not need to do an error analysis, but the number of significant digits in your answer ought reflect the accuracy of your answer.

Solution

The force between the star of mass M and the planet of mass m is given by

$$F = \frac{GMm}{R^2}$$

Assuming that $M \gg m$, then the center of mass of the system is located at the star, and therefore the orbital radius of the planet is effectively R. For a circular orbit,

$$F = m\frac{v^2}{R} = m\frac{(2\pi R/T)^2}{R} = 4\pi^2 m\frac{R}{T^2}.$$

Equating,

$$\frac{4\pi^2 R}{T^2} = \frac{GM}{R^2},$$

or the more familiar Kepler's law

$$\frac{R^3}{T^2} = \frac{GM}{4\pi^2}$$

Rearranging,

$$M = 4\pi^2\frac{GR^3}{T^2}$$

To find the mass of Fomalhaut b, M_b in terms of solar masses M_s, we simply require a ratio:

$$\frac{M_b}{M_s} = \frac{{R_b}^3/{T_b}^2}{{R_e}^3/{T_e}^2},$$

where b refers to the planet around Fomalhaut b, while e refers to Earth. Then

$$\frac{M_b}{M_s} = \left(\frac{R_b}{R_e}\right)^3\left(\frac{T_e}{T_b}\right)^2.$$

From the diagram we can obtain the ratios. Note that the 13 arcsecond notation is not needed for this computation, it simply is a comparison of the size of the object as seen from Earth. As such, seen from Earth, the orbital dust ring subtends about the same angle as does the planet Jupiter!

A quick estimate yields $R_b/R_e \approx 110$, while slightly more work is required for the periods.

The inset box is 10 times the scale of the main image.

The planet moves 1.4 AU during the two year period. Since the orbital circumference is 690 AU, that means it takes 980 years to complete an orbit. Then $T_b/T_e = 980$, and $M_b/M_s \approx 1.4$. Allowing for 10% error in *each* measurement, the results would be between 0.87 and 2.2.

Finally, after marking the quarter-final answers, it was mildly entertaining to consider the range of values obtained, from a minimum of 10^{-19} (about 10^{11} kg, or the mass of a pile of rocks some 300 meters tall) to a maximum of 10^{25} (About 1000 times greater than the mass of the universe!).

2. A ball of mass m is thrown vertically upward with a speed of v_0. The ball is subject to an air resistance force that is proportional to the velocity, $F = -kv$. The ball rises up to height of h and then returns to the starting point after some total time t_f. The acceleration of free fall is g.

   Determine the following:

   (a) An expression for the velocity as a function of time in terms of any or all of the constants.

   (b) Sketch the graph of velocity vs time. On the sketch indicate the time to rise to the highest point, t_r, and the total time of flight, t_f. You do not need to find either t_r or t_f at this point, but your graph should reflect relative positions of both.

   (c) Find the time to rise, t_r, to the highest point, h, in terms of any or all of the constants.

Solution

Two forces act on the ball: gravity, and air friction. They are not necessarily equal, so the net force, and hence an expression for the acceleration, would be given by

$$ma = -mg - kv$$

where *up* is positive. We can't simply rearrange this to solve for v, since a is a variable. We can, however, rearrange to prepare for integration, since $a = dv/dt$:

$$\frac{dv}{dt} = -\left(g + \frac{k}{m}v\right)$$

or

$$\frac{dv}{(g + \frac{k}{m}v)} = -dt.$$

Integrating both sides

$$\frac{m}{k}\ln\left(\frac{mg/k + v}{mg/k + v_0}\right) = -(t - t_0).$$

Assume t_0 is zero. Then

$$v = (mg/k + v_0)e^{-\frac{k}{m}t} - mg/k$$

Note that as $t \to \infty$ the ball will approach a constant speed of $v_t = mg/k$. Of course, it will probably hit the ground first, but if it were thrown up in the air over the edge of an infinitely deep cliff....

The ball will rise until the velocity is zero, or when

$$-\frac{v_t}{g}\ln\left(\frac{v_t}{v_t + v_0}\right) = t_r.$$

or, if you prefer,

$$t_r = \frac{v_t}{g}\ln\left(1 + \frac{v_0}{v_t}\right)$$

To find the height we need to integrate one more time. Since $v = dx/dt$ we can write

$$dx = \left((v_t + v_0)e^{-\frac{g}{v_t}t} - v_t\right)dt$$

which integrates quickly to

$$h = (v_t + v_0)\left(-\frac{v_t}{g}\right)\left(e^{-\frac{g}{v_t}t_r} - 1\right) - (v_t t_r)$$

Gluing together with the expression for t_r,

$$h = \frac{v_t v_0}{g} - \frac{{v_t}^2}{g}\ln\left(1 + \frac{v_0}{v_t}\right).$$

Showing that it reduces to $h = v_0^2/2g$ in the limit as $k \to 0$ is left as an exercise for the reader.

3. A parallel plate capacitor is made of two square parallel plates of area A, and separated by a distance $d \ll \sqrt{A}$. The capacitor is connected to a battery with potential V and allowed to fully charge. The battery is then disconnected. A square metal conducting slab also with area A but thickness $d/2$ is then fully inserted between the plates, so that it is always parallel to the plates. How much work has been done on the metal slab while it is being inserted?

Solution

The capacitance of a thin rectangular capacitor is given by

$$C = \epsilon_0 A/d$$

If you forgot this, then derive it. The electric field between the plates is given by

$$E = \frac{Q}{\epsilon_0 A d}$$

so the potential difference between the plates must be

$$V = Ed = \frac{Q}{\epsilon_0 A}$$

and the capacitance is

$$C = Q/V = \epsilon_0 A/d$$

For the original capacitor we'll call this C_0.

The energy stored in a capacitor is $U = \frac{1}{2}QV = \frac{1}{2}C{V_0}^2$, so once charged, the initial energy is

$$U_0 = \frac{1}{2}\frac{\epsilon_0 A}{d}V_0^2$$

Note, we are going to need to know the initial charge sooner or later, which is

$$Q_0 = C_0 V_0 = \frac{\epsilon_0 A V_0}{d}$$

Inserting the conducting plate can be treated two ways: it results in the creation of two capacitors, in series, each with area A but differing separation. The net charge Q_0 won't change, that's why we needed it above, but the resulting capacitance of the system will be

$$1/C_e = 1/C_1 + 1/C_2$$

For our capacitors, the only thing that changes is d, which we'll call x_1 and x_2. Then

$$1/C_e = x_1/\epsilon_0 A + x_2/\epsilon_0 A$$

which gives us

$$C_e = \frac{\epsilon_0 A}{d/2} = 2C_0$$

Since the new total separation $x_1 + x_2$ is just $d - d/2$.

The energy in the system after inserting the plate is

$$U = \frac{1}{2}Q_0 V = \frac{1}{2}Q_0^2/C_e$$

since V is *not* constant!

Finally,

$$U = \frac{1}{2}Q_0^2/2C_0 = \frac{1}{2}U_0$$

The work done *on* the plate is the same as the work done on the system, which is given by

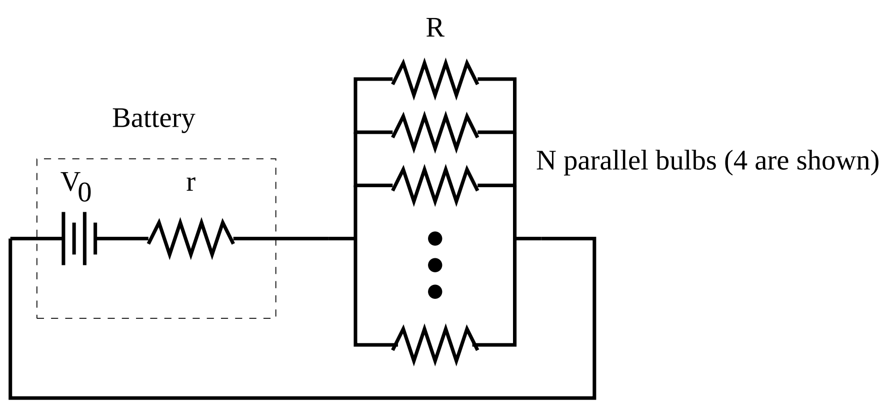
$$W = U - U_0$$

or

$$W = -\frac{1}{4}\frac{\epsilon_0 A}{d}V_0^2.$$

This means that the plate is "sucked" into the original capacitor, and we have to fight to hold it out.

4. A certain electric battery is not quite ideal. It can be thought of as a perfect cell with constant output voltage V_0 connected in series to a resistance r, but there is no way to remove this internal resistance from the battery.



The battery is connected to N identical lightbulbs in parallel. The bulbs each have a fixed resistance R, independent of the current through them.

(a) Derive an expression for the total power dissipated by the N bulbs in terms of r, R, and V_0.

(b) It is observed that the configuration dissipates more total power through the bulbs when $N = 5$ than it does for any other value of N. In terms of R, what *range* of values is possible for r?

Solution

The effective resistance of N identical resistors in parallel is

$$R_e = R/N$$

The total resistance of the circuit is the $r + R/N$.

The current drawn is then

$$I = \frac{V_0}{r + R/N}.$$

The power dissipated by the parallel resistance is given by

$$P = I^2 R_e,$$

so

$$P = \frac{{V_0}^2}{(r + R/N)^2}\frac{R}{N} = NR\frac{{V_0}^2}{(Nr + R)^2}$$

One can simply start with the fact that the maximum power is dissipated when the external resistance is equal to the internal resistance. Or, take the derivative of P with respect to R and set it equal to zero. In either case, you will get

$$r = R_e = \frac{R}{N}$$

Don't assume, however, that the answer is then $r = R/5$. Since we can't have half a bulb, we don't actually know that the power is a maximum for the circuit, only that it is the maximum configuration. It could be that a larger power output could occur with a resistance between $R/4$ and $R/5$, or maybe between $R/5$ and $R/6$.

A rough guess would be to assume

$$\frac{R}{5.5} \le r \le \frac{R}{4.5}$$

A better estimate would be to assume a quadratic function of $1/N$ with respect to P in the vicinity of $P_{\max}$ (think Taylor expansion of P), and then the important point is the halfway point between $1/N$ and $1/(N \pm 1)$, or

$$\frac{1/N + 1/(N \pm 1)}{2} = \frac{1}{2N} + \frac{1}{2(N \pm 1)} = \frac{N \pm 1/2}{N(N \pm 1)}$$

So the bounds are

$$\frac{5.5R}{30} \le r \le \frac{4.5R}{20}$$

The correct approach is to assume the power outputs at N and $N \pm 1$ are equal, and solve for the necessary value of r. Then

$$\frac{N}{(Nr + R)^2} = \frac{N \pm 1}{(Nr \pm r + R)2}$$

or

$$N^3r^2 + 2N^2(R \pm r)r + N(R \pm r)^2 = N^3r^2 + 2N^2Rr + NR^2 \pm N^2r^2 \pm 2NRr \pm R^2$$

Thankfully, we loose the cubic term

$$\pm 2N^2r^2 \pm 2NRr + Nr^2 = \pm N^2r^2 \pm 2NRr \pm R^2.$$

To save space, flip the $\pm$ to the one term that doesn't have it.

$$N^2r^2 \pm Nr^2 = R^2$$

Only positive answers have meaning, so

$$r = \frac{1}{\sqrt{N(N \pm 1)}}R$$

which give the correct bounds as

$$\frac{R}{\sqrt{30}} \leq r \leq \frac{R}{\sqrt{20}}$$

If you knew about geometric means, you might have been able to just write this down.