Four Derivations of Motional EMF

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AAPT Summer Meeting, College Park MD 29 July 2015

GOAL

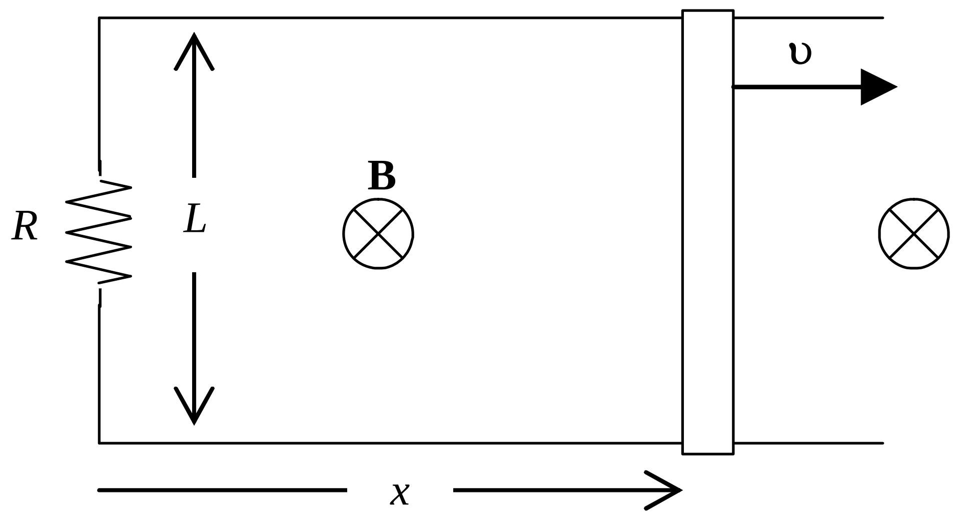
To derive and explain to students the formula

$$\varepsilon = BL\upsilon$$

for the emf generated across the ends of a straight bar of length L cutting perpendicularly at speed υ across a uniform magnetic field of magnitude B.

Derivation #1: FROM FARADAY'S LAW

Consider a conducting bar sliding frictionlessly on a U-shaped wire at constant velocity υ to the right:



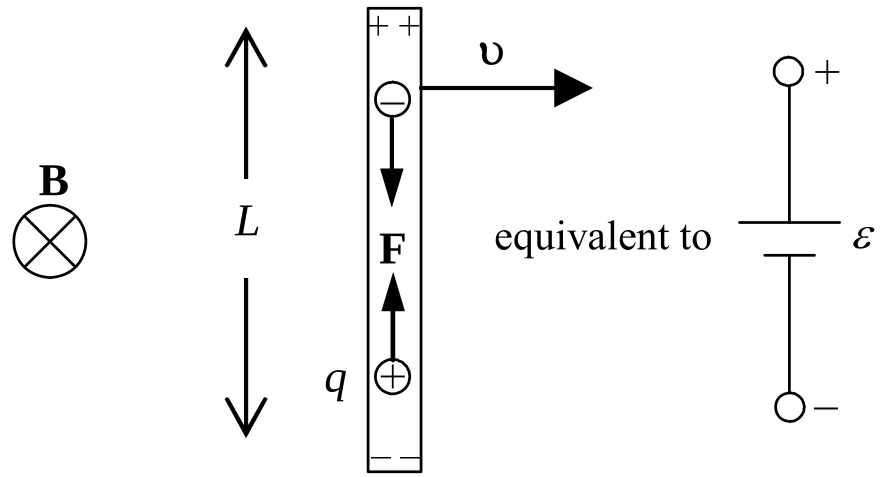
For the magnitude (ignoring signs): $\varepsilon = \dfrac{d\Phi_B}{dt} = \dfrac{d}{dt}(BLx) = BL\upsilon$

PRO: This is the fastest method of derivation.

CON: Students may erroneously believe we need a complete loop to get ε. In fact, a loop is only needed if we want the induced current $I = \varepsilon/R$.

Derivation #2: RELATION BETWEEN WORK AND POTENTIAL ENERGY

Treat the bar alone. It is composed of positive or negative free charges q that are dragged with the bar at velocity υ. Then a magnetic force acts on them of magnitude $F = q\upsilon B$ in the RHC direction:



Thus + charge collects at the top end of the bar and − charge at the bottom end (until the downward electric field created by these accumulated charges prevents further charge migration). The bar is like a battery ε whose top end is the + terminal.

worth remembering: ε increases in the direction of $\boldsymbol{\upsilon} \times \mathbf{B}$

The work required to drive charge q from one end of the bar to the other is:

$$W = FL = q\upsilon BL$$

This work goes into electric PE of the charge:

$$\Delta U = q\varepsilon$$

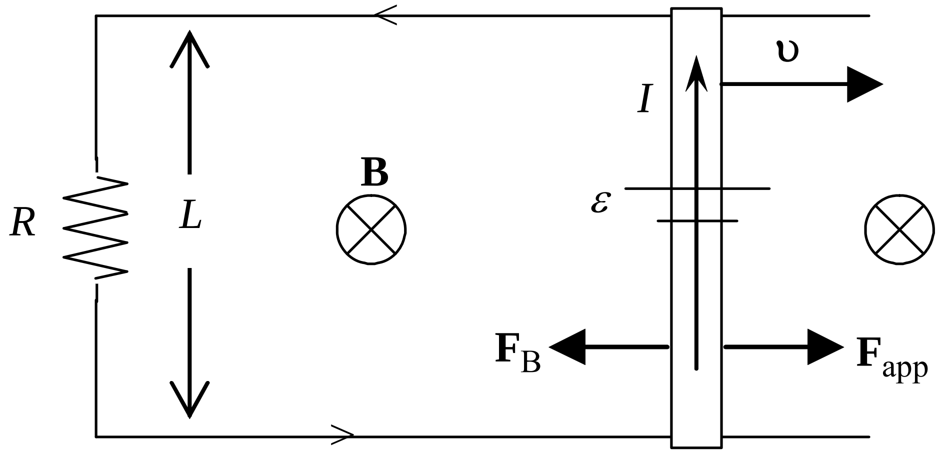
Equate the RHS of the two previous equations to reach the goal:

$$\varepsilon = \upsilon BL$$

PRO: The microscopic charge motion is now made explicit.
CON: Students may erroneously believe the gain in PE is sourced by the magnetic field. In fact, magnetic forces can never do work!

Derivation #3: SOURCE & SINK OF ENERGY

Since the bar is like a battery with + terminal on top, there is a ccw current in the loop:



Lenz's law 1st aside: This current direction agrees with Lenz's prediction.

The power we have to supply to pull the bar rightward is $P_{app} = F_{app}\upsilon$.

Our rightward applied force balances the leftward magnetic force on the current in the bar $F_B = ILB$ if the bar moves at constant velocity and thus $P_{app} = ILB\upsilon$.

Therefore our muscles are the SOURCE of energy.

This supplied energy is converted into electric energy at the rate $P_{\mathrm{E}} = I\varepsilon$.

Subsequently Joule heating of the resistor is the SINK of energy.

Now equate $P_{\mathrm{app}} = ILB\upsilon$ and $P_{\mathrm{E}} = I\varepsilon$ to once again reach the goal:

$$\varepsilon = LB\upsilon$$

Lenz's law 2nd aside: The induced current cannot be cw because then the magnetic force would accelerate the bar rather than braking it! So we see that energy conservation requires Lenz's law to hold.

Derivation #4: FROM SPECIAL RELATIVITY

The low-speed relativistic transform of the magnetic field in the lab frame gives rise to an electric field in the rest frame of the bar:

$$\mathbf{E} = \upsilon \times \mathbf{B}$$

Again ignoring signs, the magnitude is thus:

$$\varepsilon = \int \mathbf{E} \cdot d\mathbf{s} = EL = \upsilon BL$$

This is another fast method of derivation.
Also it shows how an *electrical* effect (an emf) originates from an *electric* field.

CONCLUSION

My point is not that the later derivations are *better* than the earlier ones.

Instead I think it is useful to work through *all* of them
(or just the first three methods in a nonrelativity course)
to help students understand motional emf.

In particular, Faraday's and Lenz's laws are consistent with
simple ideas about forces and energies. They're not magic!

QUESTIONS?