Damped Oscillations of a Free Piston in a Gas-Filled Cylinder

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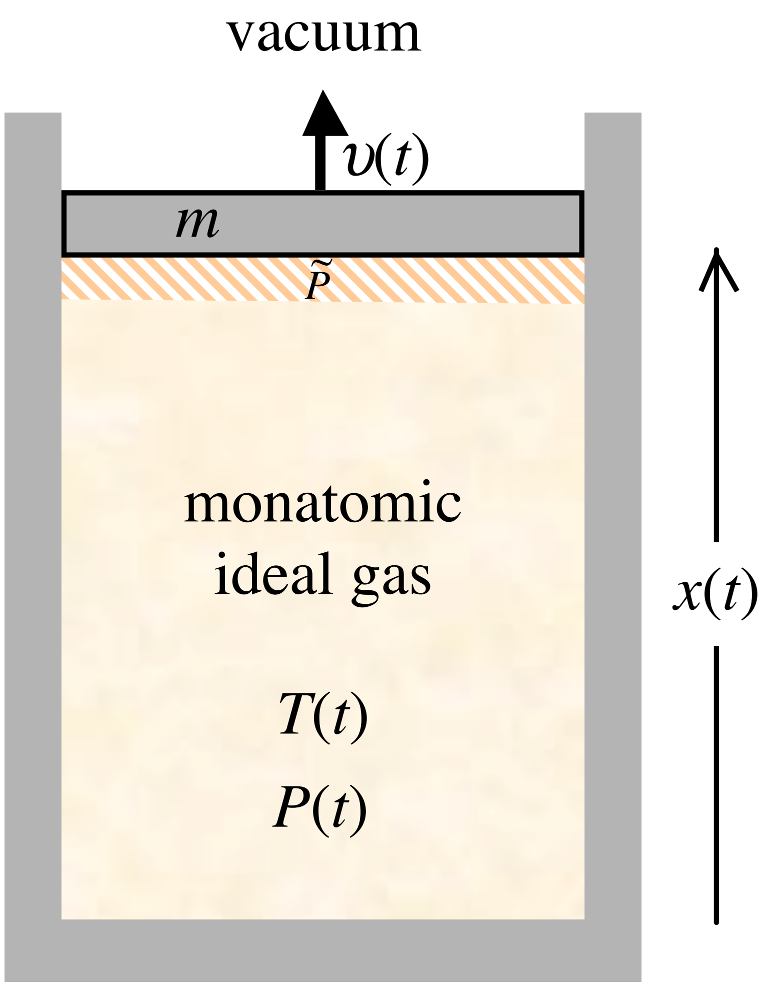
ABSTRACT

If a cylinder is capped off by a sliding piston, we have a situation analogous to a mass on a spring. With suitable idealizations[1] the mass on the spring is undamped and it will oscillate forever if initially displaced from equilibrium. With other suitable idealizations[2] will the piston similarly oscillate forever if initially displaced? No! Unlike the solid bonds inside a spring, the gas molecules are mobile and so the analog is not exact. In fact, the motion of a piston in a gas-filled cylinder is *always* damped. However, the damping is weak and so the frequency of oscillation in a Rüchardt experiment closely approximates the undamped frequency.

[1]The mass hangs vertically in vacuum from a Hookean spring attached to a rigid support.

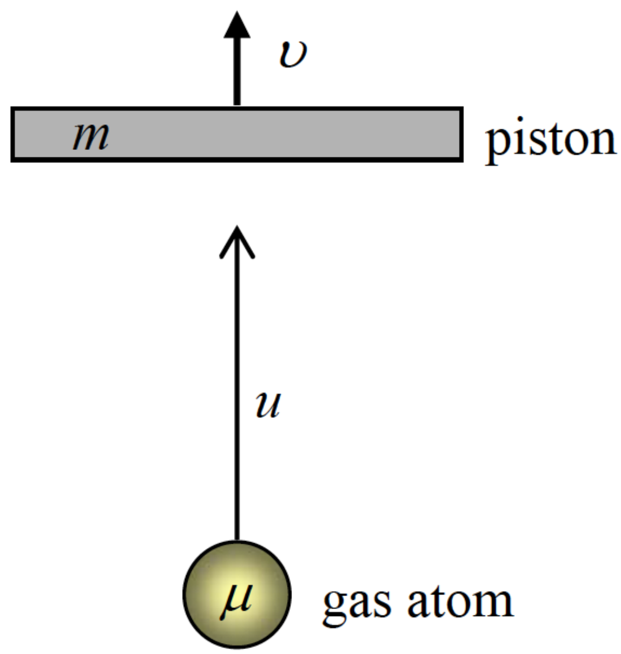
[2]The piston has no friction with the cylinder; the gas is ideal with no viscosity or turbulence; there is vacuum on the other side of the piston; and the piston and cylinder have zero thermal conductivity and heat capacity.

SETUP



The bulk of the gas has time-varying pressure P, absolute temperature T, and volume $V = Ax$. However, the gas atoms next to the piston (occupying the hatched slice of negligible volume compared to V) exert dynamic pressure $\tilde{P}$ on the piston.

DYNAMIC PRESSURE



A gas atom of mass μ and upward velocity u makes an elastic collision with the piston of mass m and upward velocity υ, where $m >> \mu$ and $u >> \upsilon$.

$$\text{kinetic theory} \Rightarrow \tilde{P} = P\left(1 - \upsilon\sqrt{\frac{8M}{\pi RT}}\right)$$

R.P. Bauman and H.L. Cockerham, "Pressure of an ideal gas on a moving piston," *Am. J. Phys.* **37**, 675 (1969)

COUPLED EQUATIONS OF MOTION

<u>Newton's second law (N2L) for the piston</u>

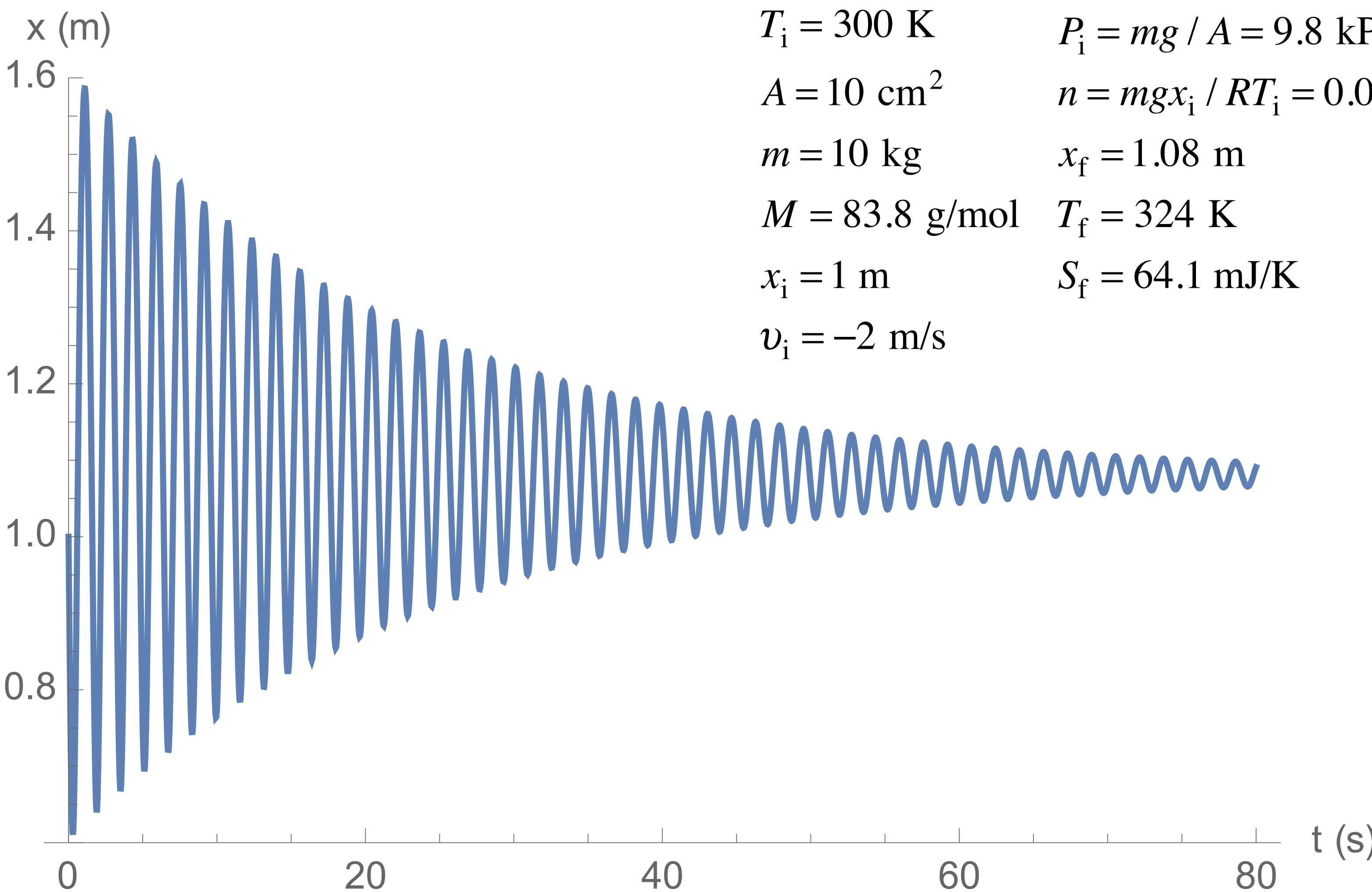
$$\tilde{P}A - mg = ma \quad \Rightarrow \quad \frac{d^2x}{dt^2} = \frac{nRT}{mx}\left(1 - \frac{dx}{dt}\sqrt{\frac{8M}{\pi RT}}\right) - g$$

<u>first law of thermodynamics (T1L) for the gas</u>

$$-\tilde{P}Adx = \frac{3}{2}nRdT \quad \Rightarrow \quad \frac{dT}{dt} = \frac{2T}{3x}\left(\frac{dx}{dt}\sqrt{\frac{8M}{\pi RT}} - 1\right)\frac{dx}{dt}$$

Solve them simultaneously for $x(t)$ and $T(t)$.

NUMERICAL SOLUTION



given

$T_\mathrm{i} = 300$ K

$A = 10$ cm^2

$m = 10$ kg

$M = 83.8$ g/mol

$x_\mathrm{i} = 1$ m

$\upsilon_\mathrm{i} = -2$ m/s

derived

$P_\mathrm{i} = mg\,/\,A = 9.8$ kPa

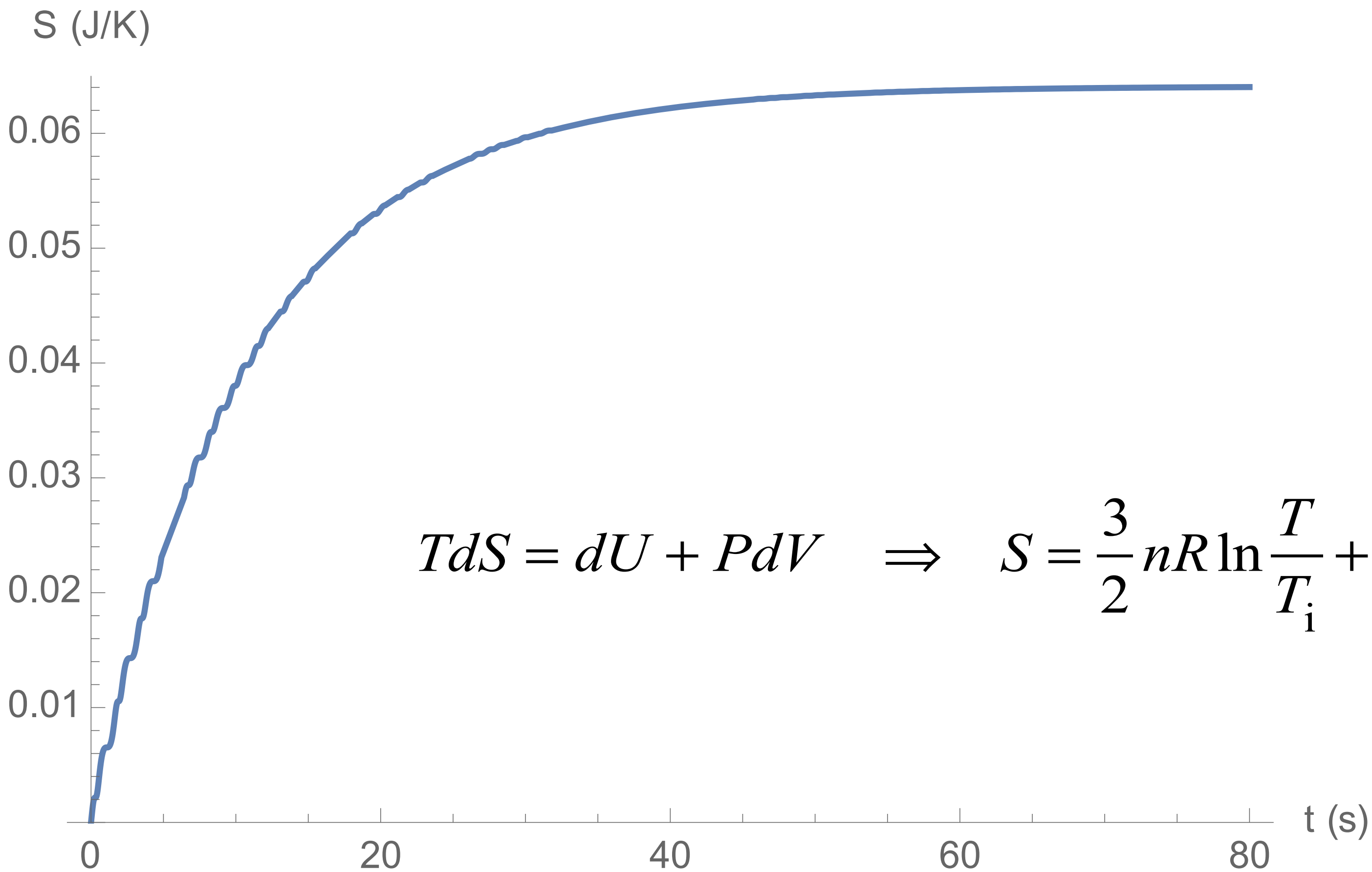
$n = mgx_\mathrm{i}\,/\,RT_\mathrm{i} = 0.0393$ mol

$x_\mathrm{f} = 1.08$ m

$T_\mathrm{f} = 324$ K

$S_\mathrm{f} = 64.1$ mJ/K

ENTROPY CHANGE



$$TdS = dU + PdV \quad \Rightarrow \quad S = \frac{3}{2} nR \ln \frac{T}{T_{\mathrm{i}}} + nR \ln \frac{x}{x_{\mathrm{i}}}$$

The increase in S confirms that the damping is irreversible.

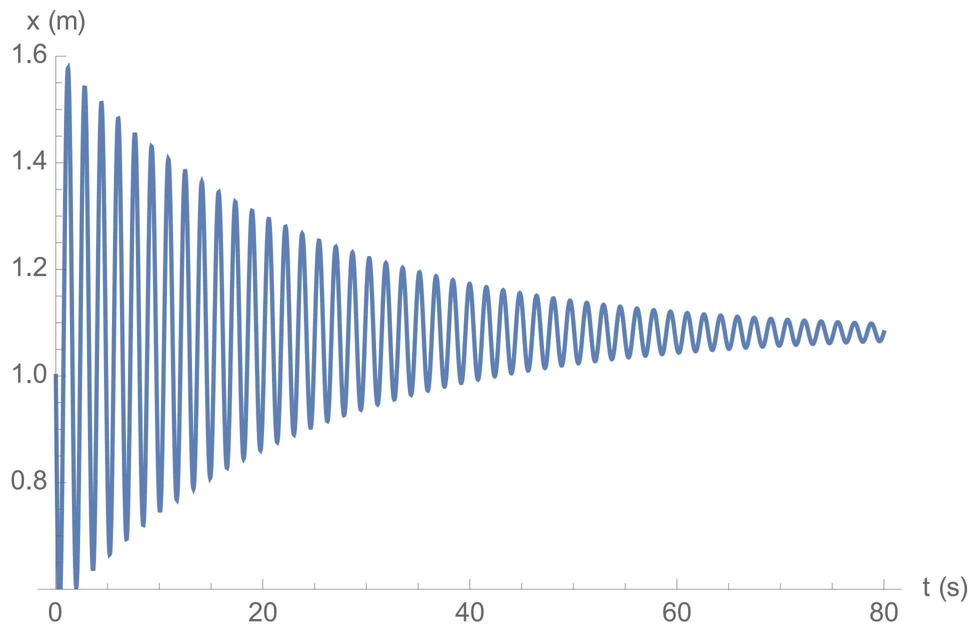
UNDERDAMPED OSCILLATOR MODEL

$$x(t) = x_{\mathrm{f}} - Xe^{-bt/2m}\sin(\omega t + \phi)$$

Rüchardt prediction $\omega = \sqrt{\dfrac{\gamma P_{\mathrm{f}} A^2}{mV_{\mathrm{f}}}} = \sqrt{\dfrac{5nRT_{\mathrm{f}}}{3mx_{\mathrm{f}}^2}} \approx 3.89\ \mathrm{rad/s}$

damping $-b\upsilon = (\tilde{P} - P)A \Rightarrow b = P_{\mathrm{f}} A\sqrt{\dfrac{8M}{\pi RT_{\mathrm{f}}}} = \dfrac{n}{x_{\mathrm{f}}}\sqrt{\dfrac{8MRT_{\mathrm{f}}}{\pi}} \approx 0.872\ \mathrm{kg/s}$

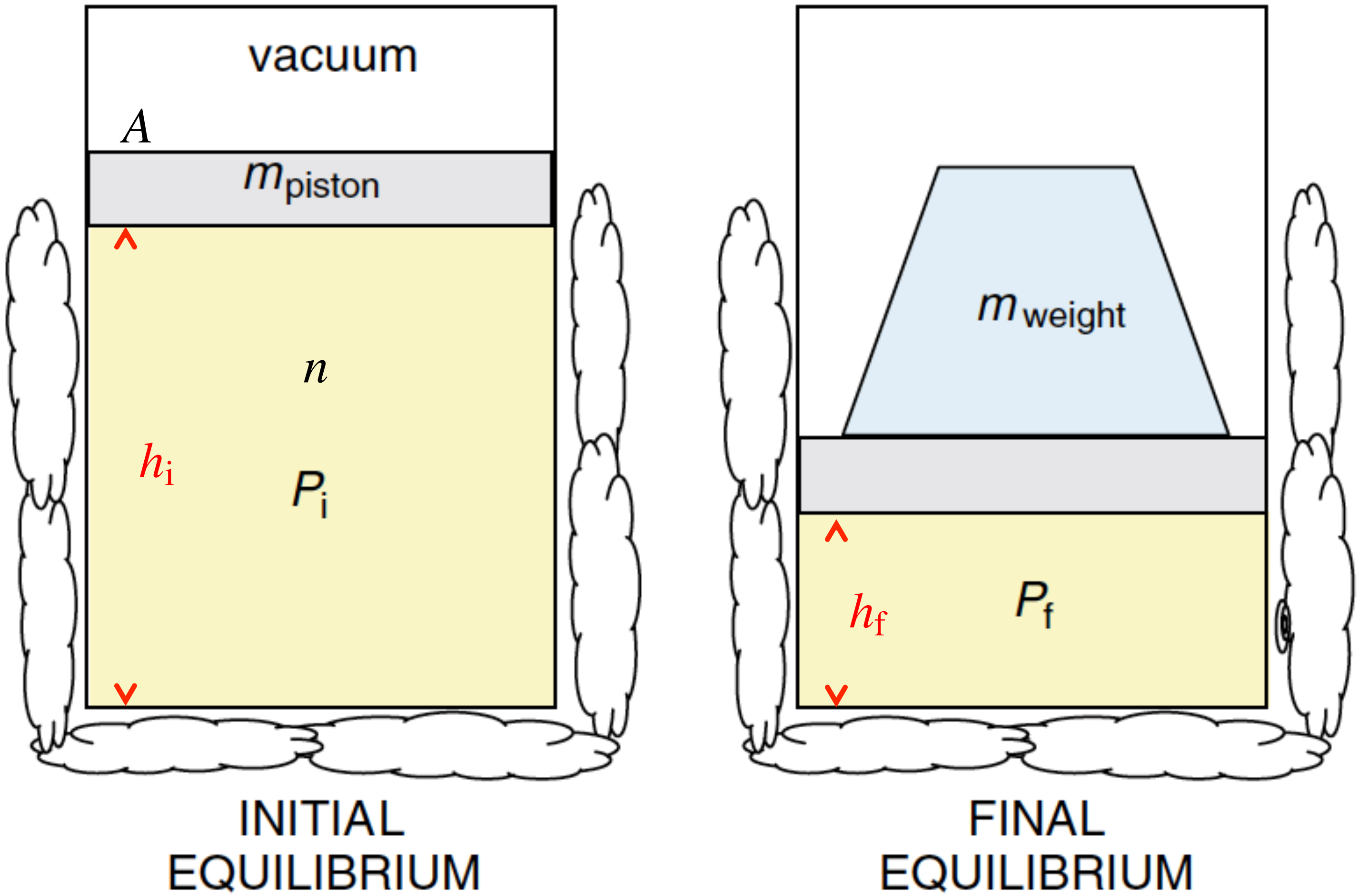
get $X = 0.522\ \mathrm{m}$ and $\phi = 0.157\ \mathrm{rad}$ from fitting x and dx/dt to x_{i} and υ_{i}



In excellent agreement with numerical solution of the coupled equations.

FINAL COMPRESSION RATIO

What is h_f after the oscillations have died away when m_{weight} is suddenly placed on the piston?



substitute N2L: $P_{\mathrm{i}}A = m_{\mathrm{piston}}g$ and $P_{\mathrm{f}}A = (m_{\mathrm{piston}} + m_{\mathrm{weight}})g$

into T1L: $\frac{3}{2}P_{\mathrm{i}}Ah_{\mathrm{i}} + (m_{\mathrm{piston}} + m_{\mathrm{weight}})gh_{\mathrm{i}} = \frac{3}{2}P_{\mathrm{f}}Ah_{\mathrm{f}} + (m_{\mathrm{piston}} + m_{\mathrm{weight}})gh_{\mathrm{f}}$

to get $\boxed{\dfrac{h_{\mathrm{f}}}{h_{\mathrm{i}}} = 1 - \dfrac{0.6}{1 + m_{\mathrm{piston}} / m_{\mathrm{weight}}}}$

so that one cannot compress the gas to less than 40% of its initial volume even if the added weight is infinite!

Contrast that with a reversible adiabatic compression described by $P_{\mathrm{i}}h_{\mathrm{i}}^{5/3} = P_{\mathrm{f}}h_{\mathrm{f}}^{5/3}$ so that $h_{\mathrm{f}} \to 0$ when the added weight (and hence the final pressure) is infinite.

RELATED PUBLICATIONS

C.E. Mungan, "Damped oscillations of a frictionless piston in an adiabatic cylinder enclosing an ideal gas," European Journal of Physics **38**, 035102 (2017).

C.E. Mungan, "Entropic damping of the motion of a piston," Physics Teacher **55**, 180 (2017).

C.E. Mungan, "Irreversible adiabatic compression of an ideal gas," Physics Teacher **41**, 450 (2003).