A brief reminder on general relativity

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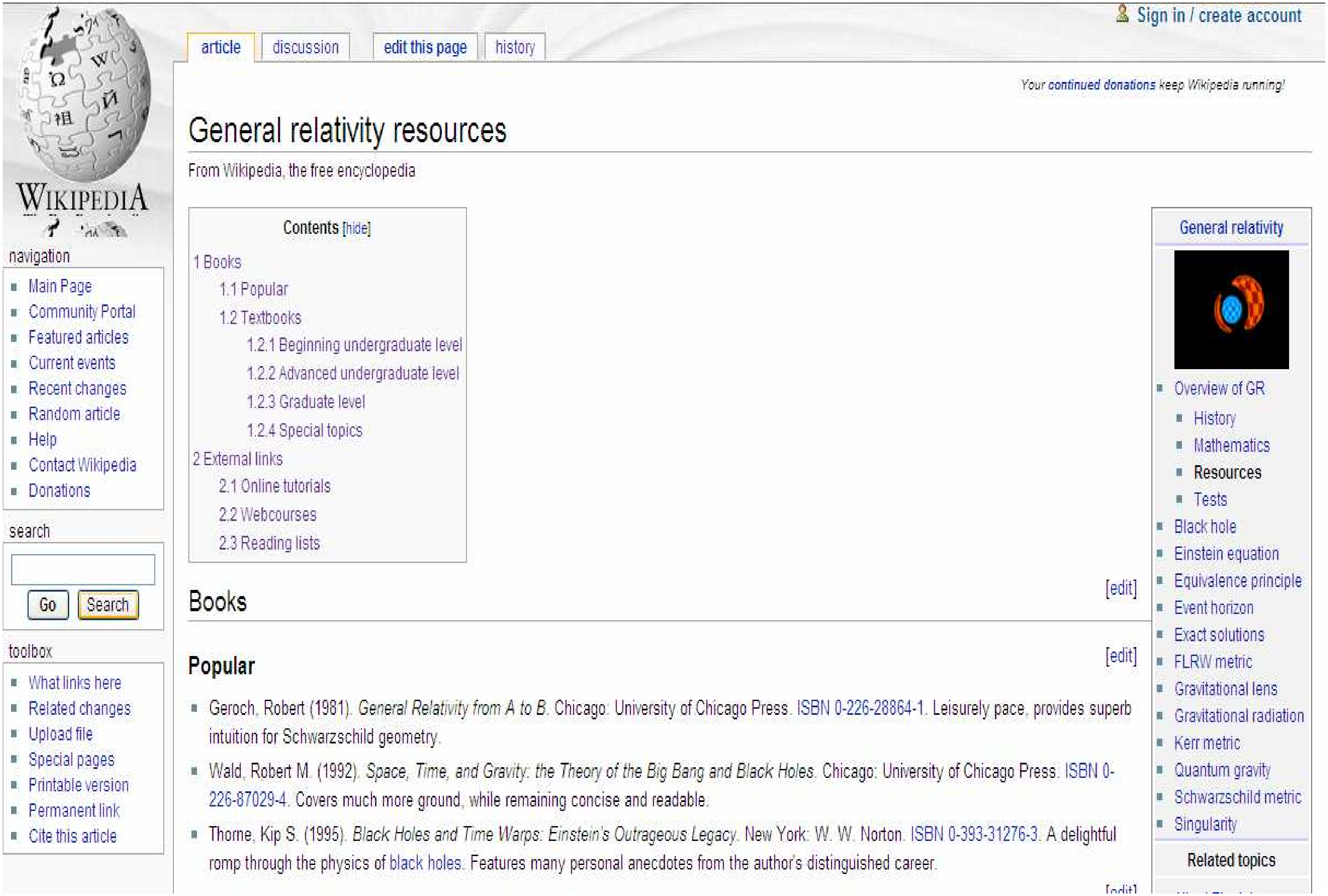
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- Geroch, Robert (1981). *General Relativity from A to B*. Chicago: University of Chicago Press. ISBN 0-226-28864-1. Leisurely pace, provides superb intuition for Schwarzschild geometry.
- Wald, Robert M. (1992). *Space, Time, and Gravity: the Theory of the Big Bang and Black Holes*. Chicago: University of Chicago Press. ISBN 0-226-87029-4. Covers much more ground, while remaining concise and readable.
- Thorne, Kip S. (1995). *Black Holes and Time Warps: Einstein's Outrageous Legacy*. New York: W. W. Norton. ISBN 0-393-31276-3. A delightful romp through the physics of black holes. Features many personal anecdotes from the author's distinguished career.

"A no-nonsense introduction to general relativity" by Sean Carroll
"Lecture notes on general relativity" by Sean Carroll (Grav. waves)

Plan:

- 1. Intro
- 2. Special relativity
- 3. Tensors
- 4. Curvature
- 5. General relativity
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Intro: GR in a nutshell

- Gravity is not a force anymore.

- Space-time is curved and that curvature accounts for what we normally think of as “gravitational forces”. More technically, space-time is a pseudo-Riemannian manifold of signature (-,+,+,+).

- The curvature of space-time is determined by matter via the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Curvature in turn determines the motion of matter.

That’s it!!

We’ll devote the rest of the talk to making sense of these statements…

Special relativity:

The speed of light is the same for all observers. This leads to setting up physics in an arena in which space and time are described by a vector space called space-time. It is like any other vector space, but it is four dimensional. We can set up coordinates on it, and to make things simple one usually chooses units where c=1.

$$
\begin{aligned}
x^0 &\equiv ct = t \\
x^1 &\equiv x \\
x^2 &\equiv y \\
x^3 &\equiv z.
\end{aligned}
$$

This vector space has a scalar product. Scalar products are bilinear forms that act on vectors and produce a number. Concretely,

$$A \cdot B \equiv \eta_{\mu\nu} A^\mu B^\nu = \sum_{\mu=1}^{4} \sum_{\mu=1}^{4} \eta_{\mu\nu} A^\mu B^\nu$$

And here we are using “Einstein’s summation convention”, a repeated index implies a summation from 1 to 4.

The matrix η determines the scalar product (usually called “metric”). And in special relativity turns out to be the “Minkowski metric”.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

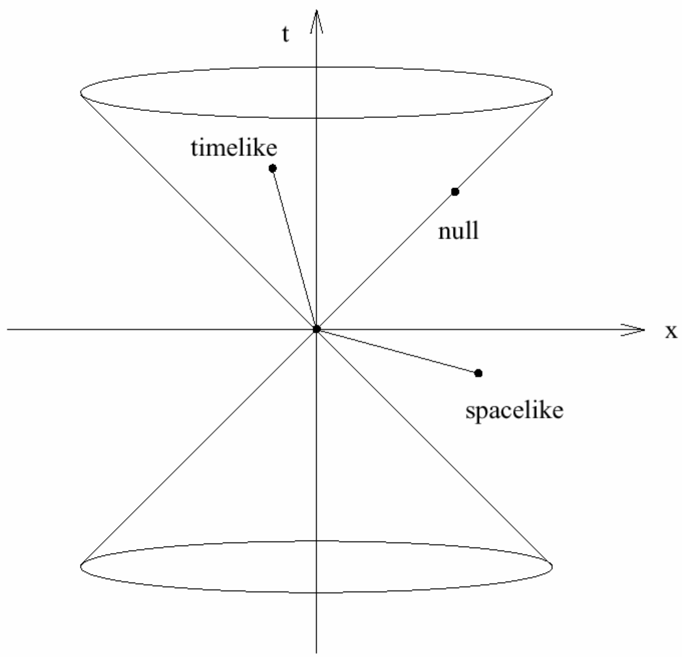
The fact that there is a minus sign has some unusual consequences. Notice: the “length” of a vector is not positive definite.

In particular the “distance” between two distinct points can be positive, negative or zero

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dx^2 + dy^2 + dz^2 \ . \end{aligned}$$

Some lingo:
A vector whose norm is negative is called “timelike”. If the norm is positive it is called “spacelike” and if the norm vanishes it is called “null”.

The minus sign is responsible for the emergence of a structure in the vector space called “light cones”



The scalar product is invariant under Lorentz transformations, which are linear maps between the coordinates (akin to the rotations of ordinary vector calculus).

If we consider the trajectory of a particle that does not move (spatially) through space-time, one has $ds^2=-dt^2<0$. This leads to define proper time as $d\tau^2=-ds^2$

Trajectories of particles can be represented as curves parameterized by the proper time $x^\mu(\tau)$. The tangent vector is called four-velocity $U^\mu = dx^\mu/d\tau$. This vector is automatically normalized $\eta_{\mu\nu}U^\mu U^\nu = -1$. A related vector is the four-momentum, defined by $p^\mu = mU^\mu$, with m a number independent of the frame called the "rest mass". The zeroth component of the vector is the energy of the particle. In the rest frame, recalling that we set c=1 we get $E=mc^2$. To get the components of the four momentum in a frame in which the particle is moving one can apply a Lorentz transformation to get,

$$p^\mu = (\gamma m, v\gamma m, 0, 0) \ ,$$

where $\gamma = 1/\sqrt{1-v^2}$. For small v, this gives $p^0 = m + \frac{1}{2}mv^2$ (what we usually think of as rest energy plus kinetic energy) and $p^1 = mv$ (what we usually think of as Newtonian momentum).

Newton's second law states that for a particle without forces acting on it,

$$\frac{dU^\mu}{d\tau} = 0, \qquad \text{or} \qquad \frac{d^2x^\mu}{d\tau^2} = 0$$

So space-time trajectories are straight lines.

Tensors:

To introduce general relativity we need to allow space-time to curve. In curved manifolds, Cartesian coordinates lose their special role. The natural thing to use are “curvilinear” coordinates. Although we all are familiar with particular examples of curvilinear coordinates, vector calculus in arbitrary coordinates requires a bit of work.

In usual Cartesian vector calculus one defines as “vector” any set of quantities that transforms under rotations like the coordinates. In curvilinear coordinates, one defines as (co)-vectors any set of quantities that transforms like the coordinate differentials. Vectors are quantities that transform like a gradient,

$$dx^{\mu'} = \frac{dx^{\mu'}}{dx^{\mu}} dx^{\mu} \qquad V^{\mu'} = \frac{dx^{\mu'}}{dx^{\mu}} V^{\mu} \qquad \frac{d\phi}{dx^{\mu'}} = \frac{dx^{\mu}}{dx^{\mu'}} \frac{d\phi}{dx^{\mu}} \qquad W_{\mu'} = \frac{dx^{\mu}}{dx^{\mu'}} W_{\mu}$$

The scalar product is given as before, as a bilinear function, but now the metric is not the Minkowski one, but the one of the curved space under consideration.

$$A \bullet B = g_{\mu\nu} A^{\mu} B^{\nu}$$

The metric also defines an isomorphism between vectors and co-vectors “raising and lowering of indices”

$$A^{\mu} = g^{\mu\nu} A_{\nu}, \quad A_{\mu} = g_{\mu\nu} A^{\nu}, \text{ with } g_{\mu\nu} \text{ and } g^{\mu\nu} \text{ inverses as matrices.}$$

A tensor is a multi-indexed object in which each index transforms as if it were in a vector.

$$S^{\mu'}{}_{\nu'\rho'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} S^{\mu}{}_{\nu\rho}$$

In Cartesian coordinates, the derivative of a vector is a tensor, $W_{\mu\nu} = \partial_{\mu} V_{\nu}$

This is due to the fact that rotations (the coordinate transformations of Cartesian coordinates) are given by matrices that do not depend on the coordinates and therefore are transparent to the partial derivative. This is not true anymore in curvilinear coordinates (remember those pages at the end of Jackson?).

In curvilinear coordinates one can construct a tensor with the derivatives of a vector, but with an additional structure called a "connection". This structure is obviously coordinate dependent (it vanishes in Cartesian). The resulting derivative is called "covariant derivative"

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}$$

So to do vector algebra we need a metric, to do vector calculus we need an additional structure on the space called a connection. This structure, like the metric is ours to choose. In *Riemannian geometry*, the structure is uniquely fixed by the metric

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu})$$

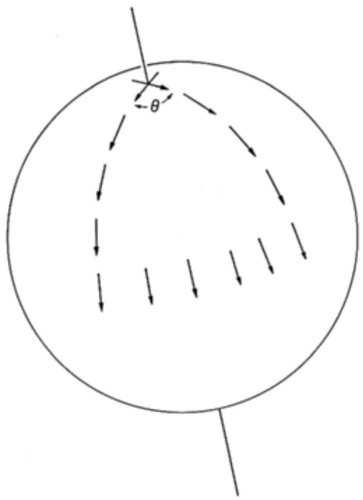
Christoffel's formula, which implies

$$\nabla_{\sigma} g_{\mu\nu} = 0 \ , \qquad \nabla_{\sigma} g^{\mu\nu} = 0$$

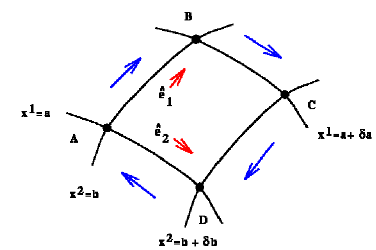
Curvature:

The presence of a connection is not a telltale sign that one has curved space (it is coordinate dependent!).

In fact, curvature cannot be characterized locally. One can always at a point choose coordinates that are locally Cartesian. To see if there is curvature one needs to "take a walk" around the manifold.



In terms of differential calculus, if one shrinks the circuit to an infinitesimal one, one still gets a contribution that is proportional to the commutator of derivatives,



$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\lambda = R_{\mu\nu\sigma}{}^{\lambda} \; V^\sigma$$

That four-indexed object is called the Riemann (or curvature) tensor. We can actually evaluate what it is from the formula for the covariant derivatives. It ends up being a non-linear combination of the metric and its derivatives up to second.

Tracing this tensor with the metric generates other tensors of relevance, the Ricci tensor and the curvature scalar.

$$R_{\mu\nu} = R_{\lambda\mu\nu}{}^{\lambda}, \qquad R = R_{\mu\nu} g^{\mu\nu}$$

General relativity:

Now that we have down the math of curved spaces, we can do physics!

We need two things:

a) A law that tells us how matter curves space-time.
b) A law that tells us how curved space-time affects usual physics.

The first of the two laws are the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

The left hand side is a combination of components of the curvature (a second order differential operator acting on the metric). The right hand side is the “stress-energy” tensor for matter. For a point particle, $T_{\mu\nu} = m U_\mu U_\nu \delta^4(x - x(\tau))$

These equations determine the metric given the matter just like $\nabla^2 \phi = 4\pi G \rho$ determines the Newtonian potential ϕ given the matter distribution ρ.

This analogy can be made precise. If you consider the metric,

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)d\vec{x}^2$$

with ϕ<<1 (weak fields), the 00 component of the Einstein equations is precisely.

The law that determines how physics behaves in curved space is called "minimal coupling" and states:
Take any flat space physics law you like written in tensor notation.
Replace the Minkowski metric wherever it appears by the curved metric.
Replace any partial derivative with covariant derivatives.
That's it!

$$\eta_{\mu\nu} \to g_{\mu\nu}$$

$$\partial_\mu \to \nabla_\mu$$

So for instance we stated that Newton's law could be written

$$\frac{dU^\mu}{d\tau} = 0, \text{ which we can write as } \frac{dx^\nu}{d\tau}\partial_\nu U^\mu = 0$$

So in curved space this would yield: $$\frac{dx^\nu}{d\tau}\nabla_\nu U^\mu = 0$$

And it turns out this is the equation of a *geodesic*, a curve of minimal length ("the closest thing to a straight line in a curved space").

If you work out the (spatial part) of the geodesic equation for the metric we discussed in the previous slide, and you assume slow speeds, you get $$\frac{d^2\vec{x}}{dt^2} = -\nabla\phi$$

Black holes:

Once one has the equations for a theory of gravitation, the first temptation is to find the gravitational field of a "particle". For this one would usually make some assumptions, like considering the field to be spherically symmetric. One would conjecture that the metric should look more or less like,

$$ds^2 = -A(r)dt^2 + B(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2)$$

(usual spheres spatially, radial dependence only). One can redefine r and get,

$$ds^2 = -A(r)dt^2 + B'(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

And substituting into the Einstein equations and assuming one is in vacuum one gets the famous Schwarzschild solution (1916),

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right)dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

(Note: you don't really need to assume time independence, it comes out as a result, this is called Birkhoff's theorem).

When r is large with respect to Gm, this metric coincides with the weak field one we considered, it therefore reproduces Newton's orbits.

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right)dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

The metric blows up at r=0 and r=2Gm. These points are called singularities. The singularity at r=0 is unavoidable. The one at r=2Gm is due to a bad choice of coordinates. Indeed, if one changes to the Kruskal coordinates,

$$u = \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} \cosh(t/4Gm)$$
$$v = \left(\frac{r}{2Gm} - 1\right)^{1/2} e^{r/4Gm} \sinh(t/4Gm).$$

The metric takes the form,

$$ds^2 = \frac{32(Gm)^3}{r} e^{-r/2Gm}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ ,$$

And is regular at r=2Gm.

If one considers the geodesic equation for $\theta=\pi/2$, one gets an effective one dimensional problem, just like for the Newtonian two-body problem,

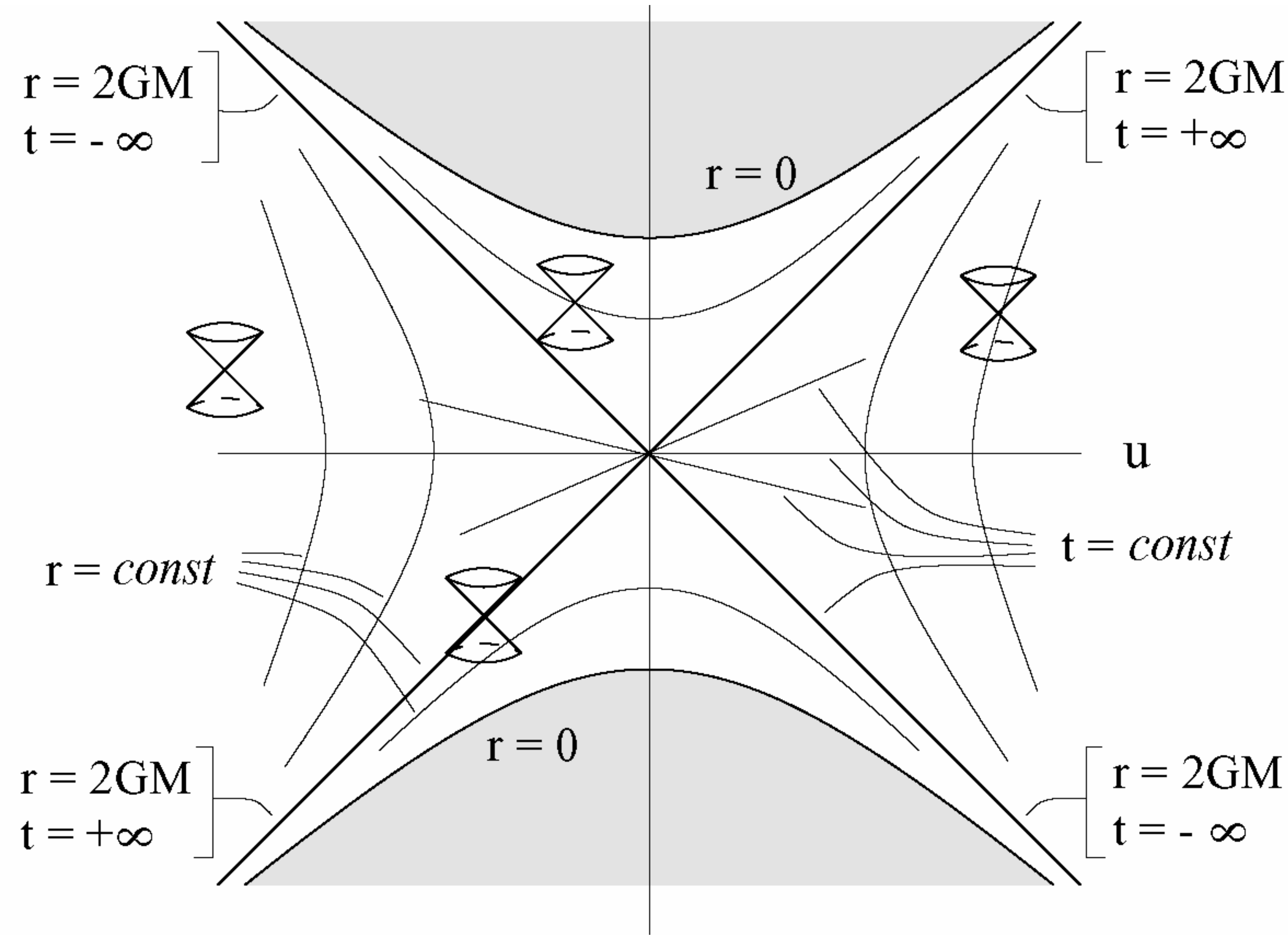
$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V(r) = \frac{1}{2}E^2 \, . \qquad V(r) = \frac{1}{2} - \frac{Gm}{r} + \frac{L^2}{2r^2} - \frac{GmL^2}{r^3} \, .$$

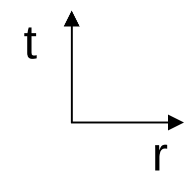
Where L is the angular momentum per unit mass of the orbiting particle.
The first three terms in the potential are the same as in Newton's theory.
The last term is a new GR contribution. That contribution is responsible for things in the weak field regime as the precession of the perihelion of Mercury.

But it also implies that for r->0, the potential goes to minus infinity, unlike in Newton's theory. That is, if you get too close to r=0, you can't escape. This is the first manifestation of a black hole.

Recall this is an "exterior" solution, it is not valid, say, within a star. For most astronomical objects, the solution stops being valid long before you "cant' escape". Black holes are objects so dense that the "no escape" region is accessible.
The point of no return is given at r=2Gm.

Another view of the issue is to consider the space-time in the Kruskal coordinates and see how they relate to more ordinary coordinates,





Gravitational waves:

In Newton's theory of gravitation changes in the sources imply instantaneous changes in the gravitational field. This is incompatible with special relativity.

In general relativity, we can get a feel for dynamical gravitational fields relatively easily if we assume the fields are weak. We will therefore consider metrics such that they are the Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| << 1 \ .$$

If one places such metric into the Einstein equations and keeps terms only up to linear order in h, one gets,

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ ,$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \ .$ and $\partial_\mu \bar{h}^\mu{}_\lambda = 0$

The latter condition is analogous to the "Lorenz gauge" in electromagnetism, in GR it corresponds to a choice of coordinates.

So indeed we see that we get a wave equation for the metric perturbations!

Since the main equation is a wave equation, we of course know how to solve it (e.g. use Green's functions),

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G\int \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})\, d^3y \ ,$$

And we see that propagation is not instantaneous, it travels at the speed of light.

What about intensity? Here we can follow a calculation totally similar to the one that is done in Maxwell's theory. The only caveat are the presence of the tensor indices. We will see they have a consequence.

One starts by assuming one has a small, far away source. Then one has x>>y and the 1/|x-y| piece can be pulled out of the integral, the y dependence in the first entry in T ignored. One then does a couple of integrations by parts and the final result is,

$$\bar{h}_{ij}(t,x) = \frac{2G}{3R}\frac{d^2 q_{ij}(t_r)}{dt^2}$$

$R=|x-y|$, $t_r=t-|x-y|$

$$q_{ij}(t) = 3\int y^i y^j T^{00}(t,y)\, d^3y$$

Final notes

- Gravity is now described by a deformation of geometry.
- How the geometry is deformed is given by the Einstein equations.
- Solutions of the equations describe well the physics of weak fields and predict new phenomena for strong fields.
- The equations have wave-like solutions like those of Maxwell theory, but the lowest contribution is quadrupolar.