THE GEOMETRY OF SPECIAL RELATIVITY

Tevian Dray
Department of Mathematics
Oregon State University

The Lorentz transformations at the heart of special relativity are just hyperbolic rotations. Special relativity itself can therefore be beautifully described in terms of "hyperbola geometry".

HYPERBOLA GEOMETRY

Euclidean distance is based on the *unit circle*, the set of points which are unit distance from the origin. Hyperbola geometry is obtained simply by using a different distance function! Measure the "squared distance" of a point $B = (x, y)$ from the origin using the definition

$$\delta^2 = x^2 - y^2$$

Then, as shown below the unit "circle" becomes the unit hyperbola

$$x^2 - y^2 = 1$$

and we further restrict ourselves to the branch with $x > 0$. If B is a point on this hyperbola, then we can *define* the hyperbolic angle β between the line from the origin to B and the (positive) x-axis to be the *Lorentzian* length $d\sigma$ of the arc of the unit hyperbola between B and the point $(1, 0)$, where $d\sigma^2 = \left|dx^2 - dy^2\right|$. We could then *define* the hyperbolic trig functions to be the coordinates (x, y) of B, that is

$$\cosh\beta = x$$
$$\sinh\beta = y$$

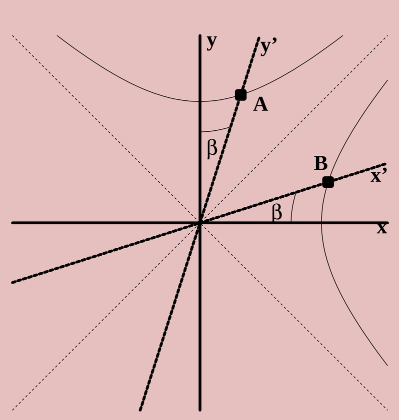
and by symmetry, the point A on this hyperbola has coordinates $(x, y) = (\sinh\beta, \cosh\beta)$; see the figure below. A little work shows that this definition is exactly the same as the standard one, namely

$$\cosh\beta = \frac{e^{\beta} + e^{-\beta}}{2}$$
$$\sinh\beta = \frac{e^{\beta} - e^{-\beta}}{2}$$

To see this, use $x^2 - y^2 = 1$ to compute

$$d\beta^2 \equiv d\sigma^2 = \left|dy^2 - dx^2\right| = \frac{dx^2}{x^2 - 1} = \frac{dy^2}{y^2 + 1}$$

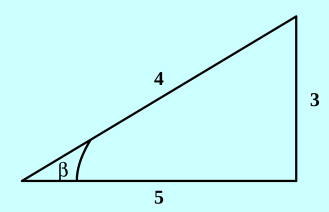
then take the square root of either expression and integrate. (The integrals are hard.) Finally, solve for x or y in terms of β.



ACKNOWLEDGMENTS

This approach to special relativity grew out of class notes for a course on *Reference Frames*, which in turn forms part of a major upper-division curriculum reform effort, the *Paradigms in Physics* project [1], begun in the Department of Physics at Oregon State University in 1997, and supported in part by NSF grants DUE–965320 and DUE–0231194. This presentation is largely excerpted from a book in preparation [2], where further details can be found, and has also appeared in [3]

TRIANGLE TRIG



We now recast ordinary triangle trig into hyperbola geometry. Suppose you know $\tanh\beta = \frac{3}{5}$, and you wish to determine $\cosh\beta$. One can of course do this algebraically, using the identity

$$\cosh^2\beta = \frac{1}{1 - \tanh^2\beta}$$

But it is easier to draw *any* triangle containing an angle whose hyperbolic tangent is $\frac{3}{5}$. In this case, the obvious choice would be the triangle shown above, with sides of 3 and 5.
What is $\cosh\beta$? Well, we first need to work out the length δ of the hypotenuse. The (hyperbolic) Pythagorean Theorem tells us that

$$5^2 - 3^2 = \delta^2$$

so δ is clearly 4. Take a good look at this 3-4-5 triangle of hyperbola geometry! Now that we know all the sides of the triangle, it is easy to see that $\cosh\beta = \frac{5}{4}$.

LORENTZ TRANSFORMATIONS

The Lorentz transformation from a moving frame (x', ct') to a frame (x, ct) at rest is given by

$$x = \gamma\left(x' + \frac{v}{c}ct'\right)$$
$$ct = \gamma\left(ct' + \frac{v}{c}x'\right)$$

We can simplify things still further. Introduce the *rapidity* β via

$$\frac{v}{c} = \tanh\beta$$

Inserting this into the expression for γ we obtain

$$\gamma = \frac{1}{\sqrt{1 - \tanh^2\beta}} = \sqrt{\frac{\cosh^2\beta}{\cosh^2\beta - \sinh^2\beta}} = \cosh\beta$$

and

$$\frac{v}{c}\gamma = \tanh\beta\cosh\beta = \sinh\beta$$

Inserting these identities into the Lorentz transformations above brings them to the remarkably simple form

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Lorentz transformations are just hyperbolic rotations!

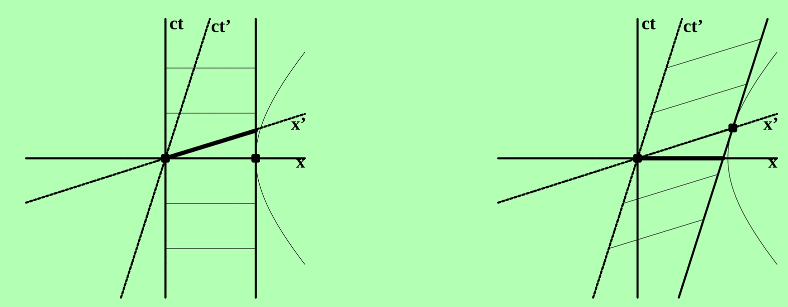
REFERENCES

[1] Corinne A. Manogue and Kenneth S. Krane, *The Oregon State University Paradigms Project: Re-envisioning the Upper Level*, Physics Today **56**, 53–58 (2003).

[2] Tevian Dray, **The Geometry of Special Relativity**, in preparation (`http://www.math.oregonstate.edu/~tevian/geometry`).

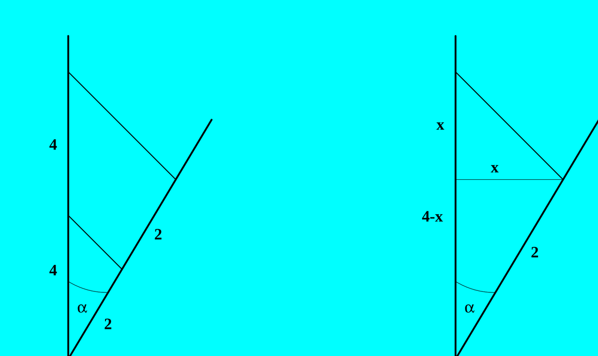
[3] Tevian Dray, *The Geometry of Special Relativity*, Physics Teacher **46**, 144–150 (2004).

LENGTH CONTRACTION



Consider first a meter stick at rest, as shown in the first sketch above. How "wide" is the *worldsheet* of the stick? The observer at rest of course measures the length of the stick by locating both ends *at the same time*, and measuring the distance between them. At $t = 0$, this corresponds to the 2 heavy dots in the sketch, one at the origin and the other on the unit hyperbola. But *all* points on the unit hyperbola are at an interval of 1 meter from the origin. The observer at rest therefore concludes, unsurprisingly, that the meter stick is 1 meter long.
How long does a moving observer think the stick is? This is just the "width" of the worldsheet *as measured by the moving observer*. This observer follows the same procedure, by locating both ends of the stick *at the same time*, and measuring the distance between them. But time now corresponds to t', not t. At $t' = 0$, this measurement corresponds to the heavy line in the sketch. Since this line fails to reach the unit hyperbola, it is clear that the moving observer measures the length of a stationary meter stick to be less than 1 meter.
To determine the exact value measured by the moving observer, use (hyperbolic) triangle trig. The heavy line in the sketch is the hypotenuse of a right triangle, whose horizontal leg measures 1 meter. If the observer is moving with speed $\frac{v}{c} = \tanh\beta$, then the length of the hypotenuse in meters must be $1/\cosh\beta$.
What if the stick is moving and the observer is at rest? This situation is shown in the second sketch above. The worldsheet now corresponds to a "rotated rectangle", indicated by the parallelograms in the sketch. The fact that the meter stick is 1 meter long in the moving frame is shown by the distance between the 2 heavy dots (along $t' = 0$), and the measurement by the observer at rest is indicated by the heavy line (along $t = 0$). Again, it is clear that the stick appears to have shrunk, since the heavy line fails to reach the unit hyperbola. Furthermore, the heavy line is again the hypotenuse of a right triangle (!), whose spacelike leg again measures 1 meter.
Thus, a moving object appears shorter by a factor $1/\cosh\beta$.

DOPPLER EFFECT



A rocket sends out flashes of light every 2 seconds in its own rest frame, which you receive every 4 seconds. How fast is it going?

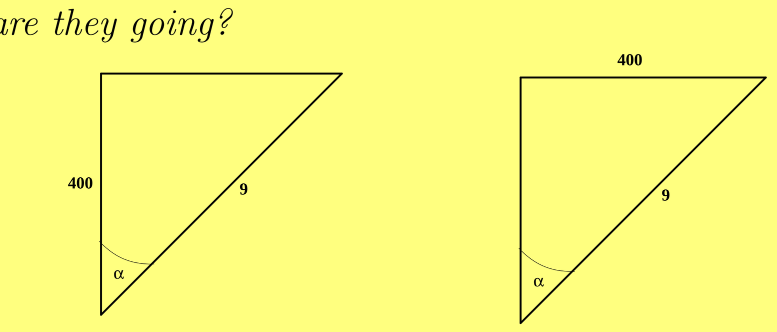
Draw a horizontal line as shown in the enlarged second drawing, so that

$$\tanh\alpha = \frac{x}{4 - x}$$
$$(4 - x)^2 - x^2 = 2^2$$

which is easily solved for $x = \frac{3}{2}$, so that $\frac{v}{c} = \tanh\alpha = \frac{3}{5}$.

COSMIC RAYS

Consider μ-mesons produced by the collision of cosmic rays with gas nuclei in the atmosphere 60 kilometers above the surface of the earth, which then move vertically downward at nearly the speed of light. The half-life before μ-mesons decay into other particles is 1.5 microseconds. Assuming it doesn't decay, how long would it take a μ-meson to reach the surface of the earth? Assuming there were no time dilation, approximately what fraction of the mesons would reach the earth without decaying? In actual fact, roughly an eighth of the mesons would reach the earth! How fast are they going?



Assume the mesons travel at the speed of light. Then it takes them

$$\frac{60 \text{ km}}{3 \times 10^8 \frac{\text{m}}{\text{s}}} = 200\ \mu\text{s}$$

to reach the earth. But $200\ \mu$s is $\frac{200}{1.5} = \frac{400}{3}$ half-lives, so only $2^{-\frac{400}{3}}$ of the mesons would reach the earth!
Since in fact an eighth reach the earth, which corresponds to 3 half-lives, time must be dilated by a factor of $\frac{400/3}{3}$, so that

$$\cosh\alpha = \frac{400}{9}$$

But, as shown in the first drawing above,

$$\frac{v}{c} = \tanh\alpha = \frac{\sqrt{400^2 - 9^2}}{400} \approx .99974684$$

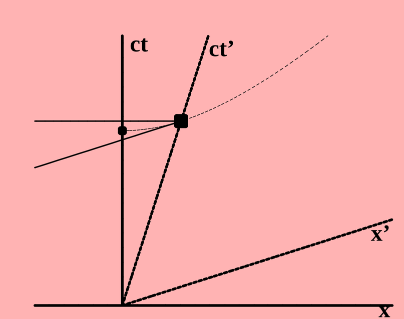
A more accurate argument would use the fact that the mesons travel 60 km in 4.5 μs (of proper time). Thus,

$$\sinh\alpha = \frac{(60 \text{ km})(1000 \frac{\text{m}}{\text{km}})}{(4.5 \times 10^{-6} \text{ s})(3 \times 10^8 \frac{\text{m}}{\text{s}})} = \frac{400}{9}$$

so that, as shown in the second drawing above,

$$\frac{v}{c} = \tanh\alpha = \frac{400}{\sqrt{400^2 + 9^2}} \approx .99974697$$

TIME DILATION



Using hyperbolic triangle trig, it is straightforward to show that moving clocks run slow by the same factor as before, namely $\cosh\beta$. Can you find the necessary right triangles for both cases in the figure?