AAPT UNITED STATES PHYSICS TEAM
AIP 2016

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 4 parts: the cover sheet (page 2), Part A (pages 3-14), Part B (pages 16-23), and several answer sheets for one of the questions in Part A (pages 25-26). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2016.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

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INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #
  Doe, Jamie
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 15, 2016.**

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

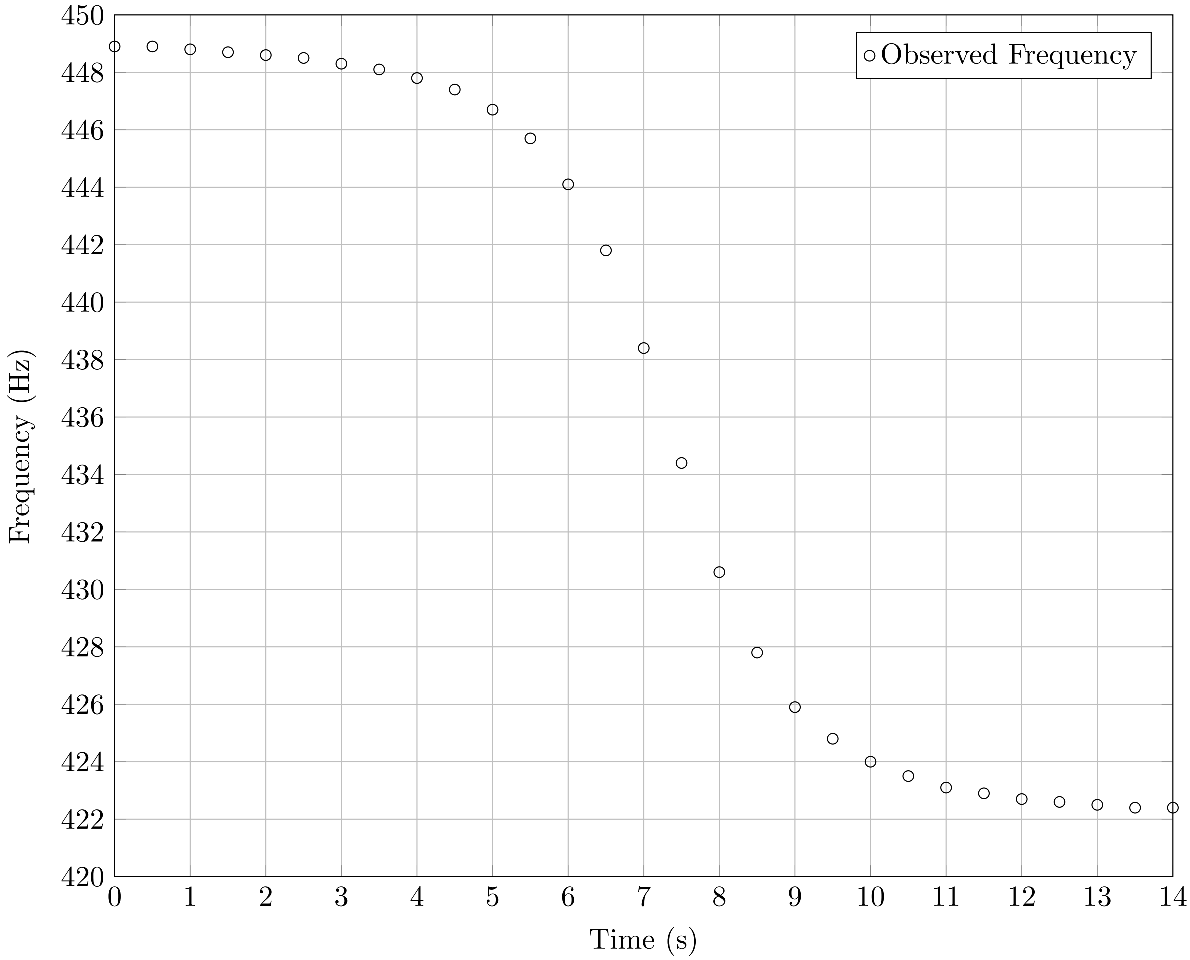
Question A1

The Doppler effect for a source moving relative to a stationary observer is described by

$$f = \frac{f_0}{1 - (v/c)\cos\theta}$$

where f is the frequency measured by the observer, f_0 is the frequency emitted by the source, v is the speed of the source, c is the wave speed, and θ is the angle between the source velocity and the line between the source and observer. (Thus $\theta = 0$ when the source is moving directly towards the observer and $\theta = \pi$ when moving directly away.)

A sound source of constant frequency travels at a constant velocity past an observer, and the observed frequency is plotted as a function of time:



The experiment happens in room temperature air, so the speed of sound is 340 m/s.

a. What is the speed of the source?

b. What is the smallest distance between the source and the observer?

Solution

(a) For θ 0 the equation reduces to

$$f_a = f_0/(1 - v/c)$$

and for $\theta \approx p$

$$f_b = f_0/(1 + v/c)$$

Read f_a and f_b off the early and late time portions of the graph and use

$$f_a/f_b = (1 + v/c)/(1 - v/c)$$

For $v \ll c$ this can be reduced to

$$f_a/f_b = 1 + 2v/c$$

(b) Let d be the (fixed) distance between the observer and the path of the source; let x be the displacement along the path, with $x = 0$ at closest approach. Then for $|x| \ll d$,

$$\cos\theta \approx \cot\theta = x/d$$

$$f = f_0/(1 - (v/c)(x/d))$$

$$f \approx f_0(1 + (v/c)(x/d))$$

Taking the time derivative, and noting that x' is simply v,

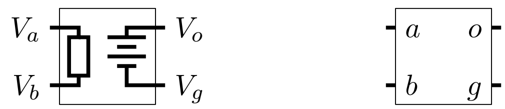
$$f' = f_0(v^2/c)d$$

From this point one can read f' off the center region of the graph and compute f_0 from the above (or, for $v \ll c$, simply use the average of the asymptotic frequencies). For $v \ll c$ a cute trick is also available: draw lines at the asymptotic values and through the central data points. The two horizontal lines are $2f_0(v/c)$ apart, so the time between the intersections is simply $2d/v$.

For this problem, $v = 10.66$ m/s, $d = 17.76$ m, and $f = 435.19$ Hz.

Question A2

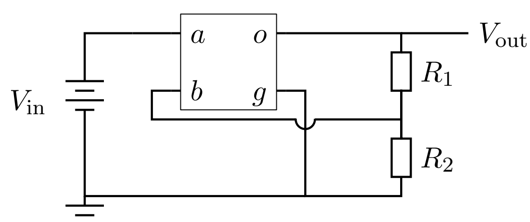
A student designs a simple integrated circuit device that has two inputs, V_a and V_b, and two outputs, V_o and V_g. The inputs are effectively connected internally to a single resistor with effectively infinite resistance. The outputs are effectively connected internally to a perfect source of emf $\mathcal{E}$. The integrated circuit is configured so that $\mathcal{E} = G(V_a - V_b)$, where G is a very large number somewhere between 10^7 and 10^9. The circuits below are chosen so that the precise value of G is unimportant. On the left is an internal schematic for the device; on the right is the symbol that is used in circuit diagrams.



Solution

The important concept is that if $\mathcal{E}$ is finite, then $V_a \approx V_b$, since G is so large. It is possible to carry around the expression for G around, and then realize that the size of G will result in it dropping out of the following expressions.

a. Consider the following circuit. $R_1 = 8.2\ k\Omega$ and $R_2 = 560\ \Omega$ are two resistors. Terminal g and the negative side of V_{in} are connected to ground, so both are at a potential of 0 volts. Determine the ratio $V_{\text{out}}/V_{\text{in}}$.



Solution

The first time we solve it we will not assume that $V_a = V_b$.

Since $V_g = 0$ (it is grounded), then $V_o = G(V_a - V_b)$. No current runs between a and b, so any current through R_1 also flows through R_2. As such

$$\frac{V_b}{R_2} = \frac{V_o}{R_1 + R_2},$$

so

$$V_o = G\left(V_i - V_o \frac{R_2}{R_1 + R_2}\right)$$

Rearrange, and

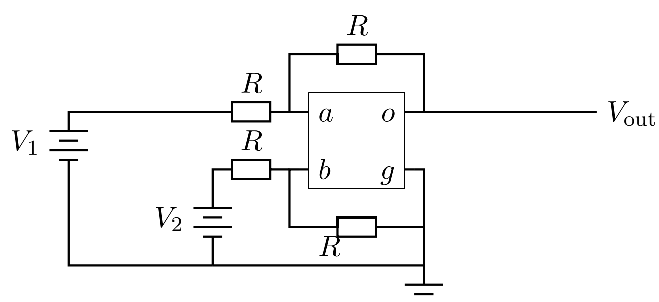
$$V_o = V_i \frac{1}{\frac{1}{G} + \frac{R_2}{R_1+R_2}}.$$

But since $G \gg R_1/R_2$, we can write

$$V_o = V_i \frac{R_1 + R_2}{R_2}.$$

This type of circuit is an amplifier with feedback.

b. Consider the following circuit. All four resistors have identical resistance R. Determine V_{out} in terms of any or all of V_1, V_2, and R.



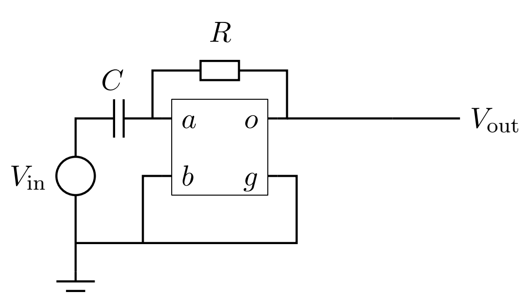
Solution

Out of laziness, we will assume that $V_a = V_b$. $V_g = 0$ because it is grounded; if current I flows through bottom resistor of picture (beneath b and g connections) then $V_2 = 2V_b$, since the voltage drop across the bottom two resistors must be equal. Similarly, the voltage drop across the top two resistors is equal, so $V_1 + V_o = 2V_a$. Combining with our initial lazy assumption,

$$V_o = V_1 - 2V_a = V_1 - V_2.$$

This type of circuit is a subtractor.

c. Consider the following circuit. The circuit has a capacitor C and a resistor R with time constant $RC = \tau$. The source on the left provides variable, but bounded voltage. Assume V_{in} is a function of time. Determine V_{out} as a function of V_{in}, and any or all of time t and τ.



Solution

Stay lazy, and $V_a = V_b = 0$.

Then $CV_i = Q$, the charge on the capacitor, and $V_o/R = I$, the current through the capacitor (no current flows through a, eh?) so

$$\frac{V_o}{R} = I = \frac{dQ}{dt} = C\frac{dV_i}{dt},$$

or

$$V_o = \tau\frac{dV_i}{dt}$$

This circuit is a differentiator.

Question A3

Throughout this problem the inertial rest frame of the rod will be referred to as the rod's frame, while the inertial frame of the cylinder will be referred to as the cylinder's frame.

A rod is traveling at a constant speed of $v = \frac{4}{5}c$ to the right relative to a hollow cylinder. The rod passes through the cylinder, and then out the other side. The left end of the rod aligns with the left end of the cylinder at time $t = 0$ and $x = 0$ in the **cylinder's frame** and time $t' = 0$ and $x' = 0$ in the **rod's frame**.

The left end of the rod aligns with the left end of the cylinder at the same time as the right end of the rod aligns with the right end of the cylinder in the **cylinder's frame**; in this reference frame the length of the cylinder is 15 m.

For the following, sketch accurate, scale diagrams of the motions of the ends of the rod and the cylinder on the graphs provided. The horizontal axis corresponds to x, the vertical axis corresponds to ct, where c is the speed of light. Both the vertical and horizontal gridlines have 5.0 meter spacing.

a. Sketch the world lines of the left end of the rod (RL), left end of the cylinder (CL), right end of the rod (RR), and right end of the cylinder (CR) in the **cylinder's frame**.

b. Do the same in the **rod's frame**.

c. On **both** diagrams clearly indicate the following four events by the letters A, B, C, and D.

   A: The left end of the rod is at the same point as the left end of the cylinder

   B: The right end of the rod is at the same point as the right end of the cylinder

   C: The left end of the rod is at the same point as the right end of the cylinder

   D: The right end of the rod is at the same point as the left end of the cylinder

d. At event B a small particle P is emitted that travels to the left at a constant speed $v_P = \frac{4}{5}c$ in the **cylinder's frame**.

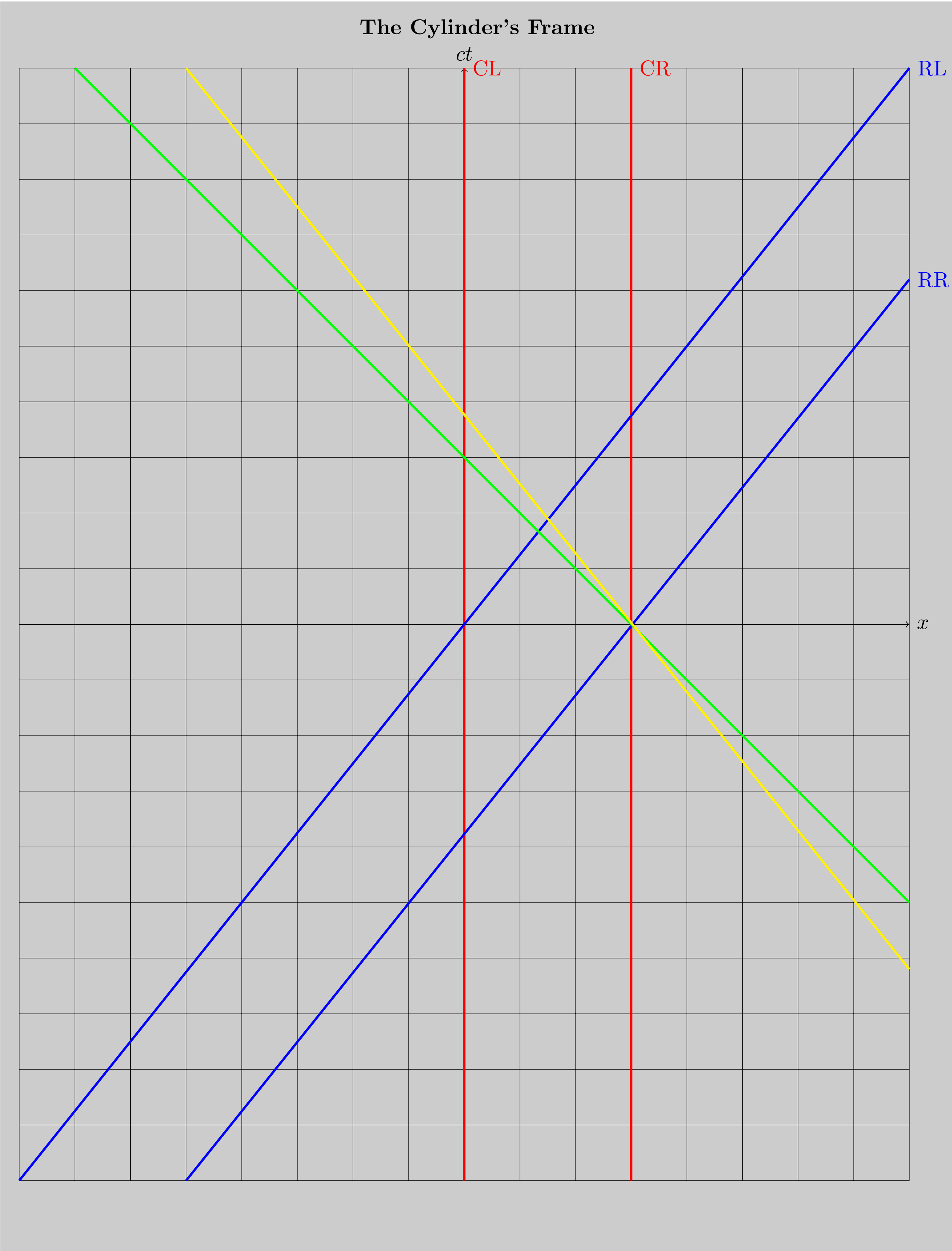
   i. Sketch the world line of P in the **cylinder's frame.**

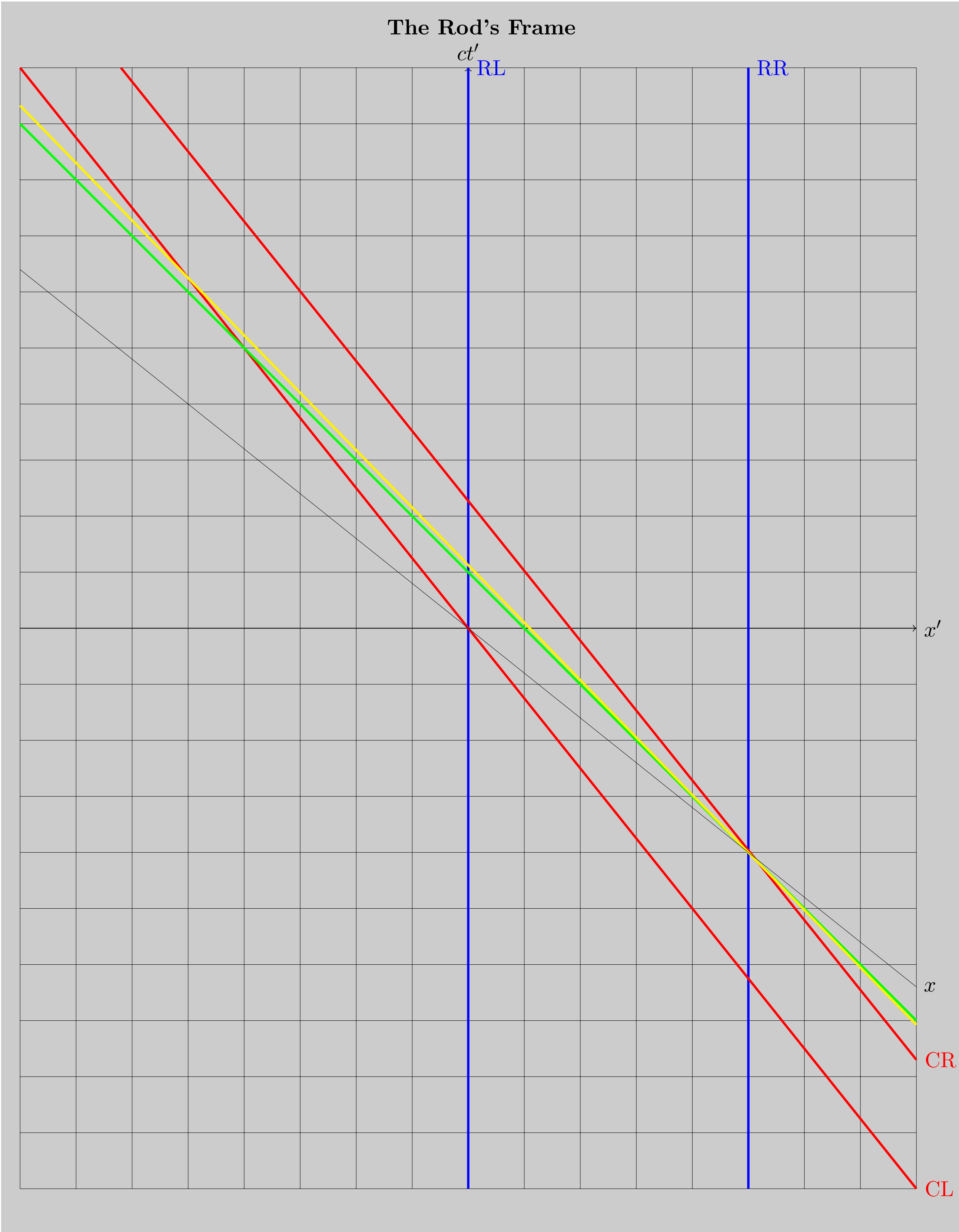
   ii. Sketch the world line of P in the **rod's frame.**

e. Now consider the following in the **cylinder's frame**. The right end of the rod stops instantaneously at event B and emits a flash of light, and the left end of the rod stops instantaneously when the light reaches it. Determine the final length of the rod after it has stopped. You can assume the rod compresses uniformly with no other deformation.

Any computation that you do must be shown on a separate sheet of paper, and not on the graphs. Graphical work that does not have supporting computation might not receive full credit.

Solution


The Cylinder's Frame
ct
CL
CR
RL
RR
x


The Rod's Frame
ct′
RL
RR
x′
x
CR
CL

Question A4

The flow of heat through a material can be described via the *thermal conductivity* κ. If the two faces of a slab of material with thermal conductivity κ, area A, and thickness d are held at temperatures differing by ΔT, the thermal power P transferred through the slab is

$$P = \frac{\kappa A \Delta T}{d}$$

A large, flat lake in the upper Midwest has a uniform depth of 5.0 meters of water that is covered by a uniform layer of 1.0 cm of ice. Cold air has moved into the region so that the upper surface of the ice is now maintained at a constant temperature of -10 °C by the cold air (an infinitely large constant temperature heat sink). The bottom of the lake remains at a fixed 4.0 °C because of contact with the earth (an infinitely large constant temperature heat source). It is reasonable to assume that heat flow is only in the vertical direction and that there is no convective motion in the water.

a. Determine the initial rate of change in ice thickness.

Solution

There are several issues that must be addressed to treat this, though some of the effects below are either minimal, or don't affect the answer.

i. Consider that the extra heat radiated into the air is only used to freeze the water on the lower surface of the ice.
ii. Consider the fact that the ice has a lower density than water, so as the water freezes it is lifting the ice above it.
iii. Consider the fact that the ice has a temperature gradient that changes as the ice layer gets thicker, so there is extra heat radiated into the air is from this cooling.
iv. Consider the fact that the water has a temperature gradient that is changing, so as the water layer gets thinner, there is extra heat radiated into the air is from this cooling.
v. Consider the fact that the pressure of the water beneath the ice might be changing.

Some of these effects we will address, others we will simply state "it is too small to change our answer".

The water right at the bottom of the ice is at the melting point: 0 C°. We will assume that there is a linear gradient of temperature in the water and a linear gradient of temperature in the ice. This is justified because the energy needed to change the temperature of the water is at most $4\text{ C}° \cdot C_{water} = 16800$ J/kg, which is small compared to the latent heat of fusion. If the temperature gradient were not uniform in the water, then there would be a net flux of heat into areas where the second derivative of the temperature with respect to height was negative, warming them up (and vice versa where the second derivative is positive). Because the energy needed to change the temperature of the water is small compared to the heat of fusion, the water would change temperature before a significant amount of ice formed, and would continue to change temperature until the second derivative of the temperature with respect to height was zero everywhere in the water. The same considerations apply to ice, so we can assume a linear temperature gradient everywhere.

The temperature gradient in the water is 4 C°/5 m. Multiplying by the conductivity, we get a power of

$$P_w = \frac{4\ \mathrm{C}^\circ}{5\ \mathrm{m}}\frac{0.57\ \mathrm{W}}{\mathrm{mC}^\circ} = 0.456\ \mathrm{W/m}^2$$

delivered through the water.

Through the ice, essentially the same calculation gives

$$P_I = \frac{10\ \mathrm{C}^\circ}{.01\ \mathrm{m}}\frac{2.2\ \mathrm{W}}{\mathrm{m}\cdot\mathrm{C}^\circ} = 2200\ \mathrm{W/m}^2$$

delivered through the ice.

The power delivered through the water is small enough to be ignored compared to the ice.

Each square meter of water directly underneath the ice loses 2200 J of energy per second. That is enough energy to freeze

$$2200\ \mathrm{W}/(330,000\ \mathrm{J/kg}) = 6.7\times 10^{-3}\ \mathrm{kg/s}$$

of water into ice. That is

$$(6.7\times 10^{-3}\ \mathrm{kg/s})/(920\ \mathrm{kg/m}^3) = 7.2\times 10^{-6}\ \mathrm{m}^3/\mathrm{s}$$

of ice formed for each square meter of ice, which means the ice is growing at a rate

$$r = 7.2\times 10^{-6}\ \mathrm{m/s} = 2.6\ \mathrm{cm/hr}$$

Each square meter of ice initially weighs 9.2 kg. A power of 2200 W is enough to lift this ice about 24 m/s against gravity. This number is so large compared to the rate that the ice does move up as the water expands as it freezes that we may ignore any work done in freezing the ice in the above calculations. We also assume that the rate of sublimation of the ice at the top of the lake is negligible.

Note: I don't think full points should be awarded unless issue 1 is addressed!

- I'm finding the actual physics analysis for (a) interesting. I think the obvious analysis observes that the bottom of the slab is at 0 C, so you simply set the latent heat equal to the heat conducted out through the slab (and neglect the heat conducted up through the much thicker water). But I think a more careful analysis would add half [1] the energy needed to cool the ice [2] to -10 degrees C. This would be a 3% correction, so not a huge deal, but you might want to figure out whether this is right and whether you want to care about it when grading.

[1] Half, because the cooling is spread across the thickness of the ice. The temperature gradient in the ice is $T' = (\delta T/d)$, where δT is the temperature difference and is the thickness. But after adding an incremental thickness dx, the entire slab does not cool by $T'dx$. Instead the top surface cools not at all and the former bottom surface cools the full $T'dx$. We are interested in the integrated temperature change over thickness, which is

$$1/2T'dx\cdot d = 1/2(\delta T/d)dx\cdot d = 1/2\delta T dx.$$

That is, the actual heat given off by cooling ice is half of what would be given off if the newly formed layer of ice cooled immediately to air temperature.

[2] But not the energy needed to cool the water – even though the water does cool as the ice boundary moves downward, at initial conditions the heat given off is not appreciably conducted to the boundary. I think.

b. Assuming the air stays at the same temperature for a long time, find the equilibrium thickness of the ice.

Solution

This part can be answered without answering any of the previous parts.

If the water began with the initial 1.0 cm of ice melted, the depth would be 5.0092 m instead of 5.00 m and the eventual equilibrium depth of ice would be unchanged, so we will solve using the variable $h_0 = 5.0092$ m for the initial depth of the lake, all in liquid water.

The ice will stop getting thicker when the energy flux through the water equals that through the ice, that is, when

$$\frac{\Delta T_{water}}{h_{water}}\kappa_{water} = \frac{\Delta T_{ice}}{h_{ice}}\kappa_{ice}$$

If the thickness of the water is h_{water}, the amount of water that has frozen into ice had a thickness of the $h_0 - h_{water}$. Ice is less dense than water and therefore more thick for the same mass, so the thickness of the ice obeys

$$h_{ice}\rho_{ice} = (h_0 - h_{water})\rho_{water}$$

to keep the mass of ice equal to the mass frozen out of the water. This implies

$$h_{water} = \frac{h_0\rho_{water} - h_{ice}\rho_{ice}}{\rho_{water}}.$$

Plugging this into the previous expression for equilibrium we have

$$\frac{\Delta T_{water}\kappa_{water}\rho_{water}}{h_0\rho_{water} - h_{ice}\rho_{ice}} = \frac{\Delta T_{ice}\kappa_{ice}}{h_{ice}}$$

which can be solved for h_{ice} to give

$$h_{ice} = h_0\frac{\Delta T_{ice}\kappa_{ice}\rho_{water}}{\Delta T_{water}\kappa_{water}\rho_{water} + \Delta T_{ice}\kappa_{ice}\rho_{ice}}$$

We have all the relevant values, plugging them in we obtain

$$h_{ice} = 4.89 \text{ m}$$

c. Explain why convective motion can be ignored in the water.

Solution

Convection occurs when boiling a pot of water because the hot water at the bottom of the pot is lower density than the colder water higher up. This means gravitational energy can be released when that hot, low-density water rises and cold, high-density water falls. When the hot water rises, it releases heat, cools, gets denser, and falls back down again. This cycle is convection. This relies on water having lower density the hotter it is.

In the lake, the scenario is different because although the bottom of the lake is warmer, the water there is not more dense. Water gets denser the colder it is up to 4 C°, but below that temperature it gets less dense. Thus, the water at the bottom of the lake, though warmer, is more dense than the water above it. That means there is nothing to drive convection in the water because moving some water around vertically would not release any gravitational potential energy. The water is already in mechanical equilibrium and so does not move.

Some important quantities for this problem:

Specific heat capacity of water	C_{water}	4200 J/(kg · C°)
Specific heat capacity of ice	C_{ice}	2100 J/(kg · C°)
Thermal conductivity of water	κ_{water}	0.57 W/(m · C°)
Thermal conductivity of ice	κ_{ice}	2.2 W/(m · C°)
Latent heat of fusion for water	L_{f}	330,000 J/kg
Density of water	ρ_{water}	999 kg/m^3
Density of ice	ρ_{ice}	920 kg/m^3

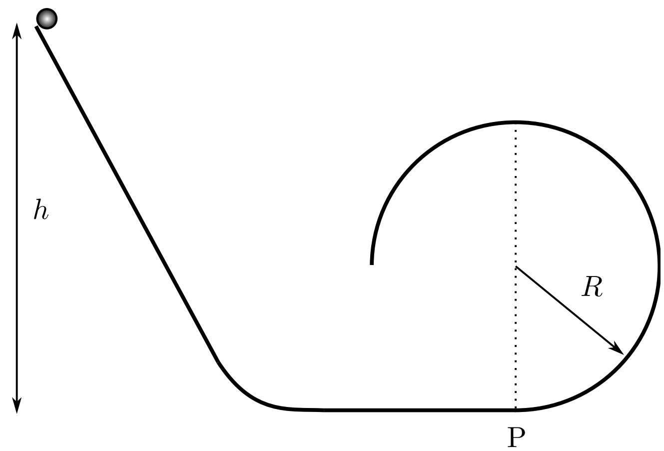
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

A uniform solid spherical ball starts from rest on a loop-the-loop track. It rolls without slipping along the track. However, it does not have enough speed to make it to the top of the loop. From what height h would the ball need to start in order to land at point P directly underneath the top of the loop? Express your answer in terms of R, the radius of the loop. Assume that the radius of the ball is very small compared to the radius of the loop, and that there are no energy losses due to friction.



Solution

Fix the coordinate system on P.

The rotational inertia of the ball is given by $I = \beta m r^2$, where $\beta = 2/5$.

The initial potential energy of the ball is mgh.

Assume the ball leaves at an angle θ away from the vertical. Then the x and y coordinates are

$$x = R \sin\theta$$

and

$$y = R(1 + \cos\theta)$$

The kinetic energy of the ball when it leaves the loop is

$$K = mg(h - y) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

but since the ball rolls without slipping,

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m(1+\beta)v^2$$

Call the speed of the ball when it leaves the loop v.

The x and y components of the velocity when the ball leaves the loop are given by

$$v_x = v\cos\theta$$

and

$$v_y = v\sin\theta.$$

Yes, I was lazy with signs.

The ball is in a free fall for a time t where

$$y = \frac{1}{2}gt^2 - v_y t$$

and it moves horizontally a distance

$$x = v_x t$$

Substitute from the equation for x above and

$$R\sin\theta = vt\cos\theta$$

or

$$t = \frac{R}{v}\frac{\sin\theta}{\cos\theta}$$

so

$$R + R\cos\theta = \frac{1}{2}g\left(\frac{R}{v}\frac{\sin\theta}{\cos\theta}\right)^2 - v\sin\theta\frac{R}{v}\frac{\sin\theta}{\cos\theta},$$

or

$$(1+\cos\theta)\cos\theta + \sin^2\theta = \frac{gR}{2v^2}\frac{\sin^2\theta}{\cos\theta}$$

or

$$1+\cos\theta = \frac{gR}{2v^2}\frac{\sin^2\theta}{\cos\theta}$$

The ball leaves the surface when the normal component of the force of the loop on the ball just drops to zero. This happens when

$$mg\cos\theta = m\frac{v^2}{R}$$

This means

$$\frac{v^2}{gR} = \cos\theta$$

so

$$1+\cos\theta = \frac{1}{2}\frac{1-\cos^2\theta}{\cos^2\theta}$$

or

$$2\cos^2\theta = 1-\cos\theta.$$

This can be solved as a quadratic, with solutions

$$\cos\theta = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1}{4}\pm\frac{3}{4}$$

Only the positive answer of $\cos\theta = 1/2$ is relevant to the problem here (though the negative answer is still physical!)

Combine with

$$mg(h-y) = \frac{1}{2}m(1+\beta)v^2$$

from above, and the expression for circular motion

$$mg\cos\theta = m\frac{v^2}{R}$$

and get

$$\frac{h}{R} = 1 + \left(\frac{1}{2}(1+\beta) + 1\right)\cos\theta = 1 + \left(\frac{1}{2}\left(1 + \frac{2}{5}\right) + 1\right)\frac{1}{2} = 1.85$$

Question B2

a. A spherical region of space of radius R has a uniform charge density and total charge $+Q$. An electron of charge $-e$ is free to move inside or outside the sphere, under the influence of the charge density alone. For this first part ignore radiation effects.

i. Consider a circular orbit for the electron where $r < R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

Solution

Apply Gauss' Law to find the E field inside the sphere:

$$\begin{aligned}
\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \frac{Q_{enc}}{\epsilon_0}, \\
\oint E dA &= \frac{Q}{\epsilon_0} \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}, \\
E 4\pi r^2 &= \frac{Q}{\epsilon_0} \frac{r^3}{R^3}, \\
E &= \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}.
\end{aligned}$$

Apply circular motion physics,

$$\begin{aligned}
m\frac{4\pi^2 r}{T^2} &= eE, \\
m\frac{4\pi^2 r}{T^2} &= e\frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}, \\
T^2 &= \frac{16\pi^3\epsilon_0 m R^3}{eQ}, \\
T &= 2\pi\sqrt{\frac{4\pi\epsilon_0 m R^3}{eQ}}.
\end{aligned}$$

Yes, it is independent of r.

ii. Consider a circular orbit for the electron where $r > R$. Determine the period of the orbit T in terms of any or all of r, R, Q, e, and any necessary fundamental constants.

Solution

Apply Gauss' Law to find the E field outside the sphere:

$$\begin{aligned}
\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \frac{Q_{enc}}{\epsilon_0}, \\
\oint E dA &= \frac{Q}{\epsilon_0}, \\
E 4\pi r^2 &= \frac{Q}{\epsilon_0},
\end{aligned}$$

$$E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}.$$

Apply circular motion physics,

$$\begin{aligned} m\frac{4\pi^2 r}{T^2} &= eE, \\ m\frac{4\pi^2 r}{T^2} &= e\frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}, \\ T^2 &= \frac{16\pi^3\epsilon_0 m r^3}{eQ}, \\ T &= 2\pi\sqrt{\frac{4\pi\epsilon_0 m r^3}{eQ}}. \end{aligned}$$

Yes, it is dependent of r. You will hopefully recognize Kepler's law. It is okay to start from a statement like "outside a spherically symmetric charge distribution it is possible to treat the distribution as a point charge."

iii. Assume the electron starts at rest at $r = 2R$. Determine the speed of the electron when it passes through the center in terms of any or all of R, Q, e, and any necessary fundamental constants.

Solution

Use the results of above and find the potential difference between the center and $r = 2R$.

$$\begin{aligned} \Delta V &= -\int_{2R}^{0} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}, \\ &= \int_{2R}^{R} \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} + \int_{R}^{0} \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}, \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{-1}{2R} - \frac{-1}{R} + \frac{R^2}{2R^3}\right), \\ &= \frac{Q}{4\pi\epsilon_0 R} \end{aligned}$$

Then use work-energy,

$$\begin{aligned} v &= \sqrt{\frac{2}{m}e\Delta V}, \\ &= \sqrt{\frac{2eQ}{4\pi\epsilon_0 m R}}. \end{aligned}$$

b. Accelerating charges radiate. The total power P radiated by charge q with acceleration a is given by

$$P = C\xi a^n$$

where C is a dimensionless numerical constant (which is equal to $1/6\pi$), ξ is a physical constant that is a function only of the charge q, the speed of light c, and the permittivity of free space ϵ_0, and n is a dimensionless constant. Determine ξ and n.

Solution

Dimensional analysis is the way to go. a has dimensions of $[\mathrm{L}]/[\mathrm{T}]^2$, P has dimensions of $[\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3$, c has dimensions of $[\mathrm{L}]/[\mathrm{T}]$, q has dimensions of $[\mathrm{C}]$, and ϵ_0 has dimensions of $[\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3$.

Set up an equation such as

$$P = a^\alpha c^\beta {\epsilon_0}^\gamma q^\delta$$

or

$$\left([\mathrm{M}][\mathrm{L}]^2/[\mathrm{T}]^3\right) = \left([\mathrm{L}]/[\mathrm{T}]^2\right)^\alpha \left([\mathrm{L}]/[\mathrm{T}]\right)^\beta \left([\mathrm{C}]^2[\mathrm{T}]^2/[\mathrm{M}][\mathrm{L}]^3\right)^\gamma \left([\mathrm{C}]\right)^\delta$$

Charge is only balanced if $\gamma = -2\delta$. Mass is only balanced if $\gamma = -1$. Similar expressions exist for length and time, yielding

$$P = \frac{1}{6\pi} a^2 c^{-3} {\epsilon_0}^{-1} q^2$$

c. Consider the electron in the first part, except now take into account radiation. Assume that the orbit remains circular and the orbital radius r changes by an amount $|\Delta r| \ll r$.

i. Consider a circular orbit for the electron where $r < R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

Solution

The energy radiated away is given by

$$\Delta E = -PT,$$

where T is determined in the previous sections.

It is possible to compute the actual energy of each orbit, and it is fairly trivial to do for regions $r > R$, but perhaps there is an easier, more entertaining way. Consider

$$\Delta E = \Delta K + \Delta U$$

and for small changes in r,

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies (correctly) that the potential energy increases with increasing r.

$$\frac{\Delta K}{\Delta r} \approx \frac{d}{dr}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}\frac{d}{dr}\left|r\frac{mv^2}{r}\right|$$

but $mv^2/r = F$, so

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF|$$

and then

$$\frac{\Delta K}{\Delta r} \approx \frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}.$$

This implies (correctly) that the kinetic energy increases with increasing r. Not a surprise, since this region acts similar to a multidimensional simple harmonic oscillator. Combine, and

$$\frac{\Delta E}{\Delta r} \approx 2\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = 2ma$$

Finally,

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right)\left(\frac{1}{2ma}\right)$$

This can be simplified, so

$$\begin{aligned}
\Delta r &= -\left(\frac{1}{12\pi}\frac{a}{mc^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right), \\
&= -\left(\frac{1}{6}\frac{e^2}{m^2c^3\epsilon_0}\right)\left(\sqrt{\frac{4\pi\epsilon_0 mR^3}{eQ}}\right)\left(\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3}\right), \\
&= -\frac{1}{6}\sqrt{\frac{e^5Q}{4\pi{\epsilon_0}^3R(mc^2)^3}}\frac{r}{R}
\end{aligned}$$

You might want to group these in terms of dimensionless groupings:

$$\Delta r = -\frac{2}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}}r$$

ii. Consider a circular orbit for the electron where $r > R$. Determine the change in the orbital radius Δr during one orbit in terms of any or all of r, R, Q, e, and any necessary fundamental constants. Be very specific about the sign of Δr.

Solution

Pick up where we left off, and

$$\frac{\Delta U}{\Delta r} \approx -F = \frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}.$$

This implies (correctly) that the potential energy increases with increasing r.

$$\frac{\Delta K}{\Delta r} \approx \frac{1}{2}\frac{d}{dr}|rF|$$

and so

$$\frac{\Delta K}{\Delta r} \approx -\frac{eQ}{8\pi\epsilon_0}\frac{1}{r^2}.$$

This implies (correctly) that the kinetic energy decreases with increasing r. Combine, and

$$\frac{\Delta E}{\Delta r} \approx \frac{1}{2}\frac{eQ}{4\pi\epsilon_0}\frac{r}{R^3} = \frac{ma}{2}$$

Follow the same type of substitutions as before, and

$$\Delta r = -\left(\frac{1}{6\pi}\frac{a^2}{c^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mr^3}{eQ}}\right)\left(\frac{2}{ma}\right)$$

This can be simplified, so

$$\begin{aligned}
\Delta r &= -\left(\frac{1}{3\pi}\frac{a}{mc^3\epsilon_0}e^2\right)\left(2\pi\sqrt{\frac{4\pi\epsilon_0 mr^3}{eQ}}\right), \\
&= -\left(\frac{2}{3}\frac{e^2}{m^2c^3\epsilon_0}\right)\left(\sqrt{\frac{4\pi\epsilon_0 mr^3}{eQ}}\right)\left(\frac{eQ}{4\pi\epsilon_0}\frac{1}{r^2}\right), \\
&= -\frac{1}{3}\sqrt{\frac{e^5 Q}{4\pi\epsilon_0{}^3 r(mc^2)^3}}
\end{aligned}$$

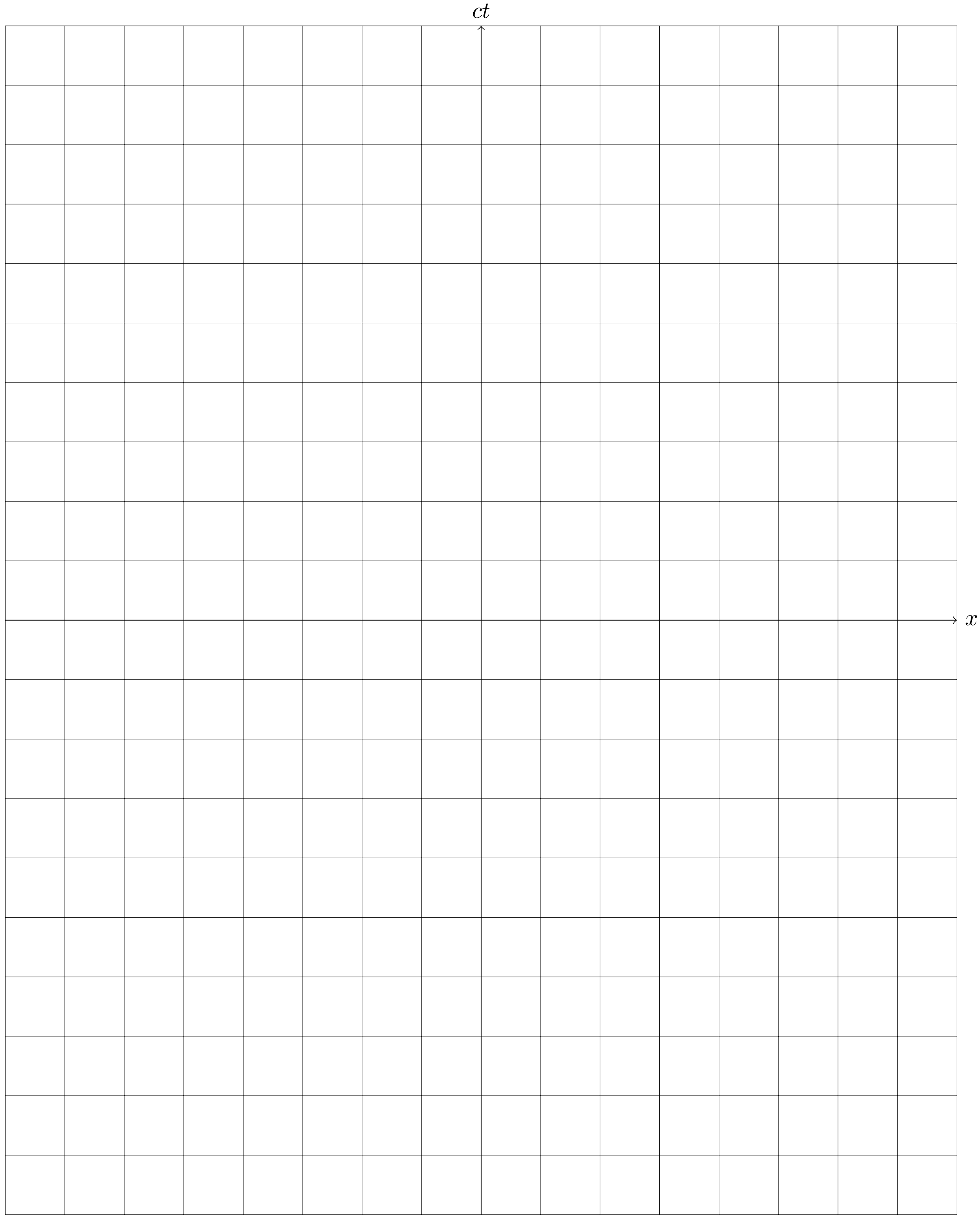
You might want to group these in terms of dimensionless groupings:

$$\Delta r = -\frac{4}{3}\left(\frac{e^2}{4\pi\epsilon_0 Rmc^2}\right)\sqrt{\frac{eQ}{4\pi\epsilon_0 Rmc^2}}\frac{R^2}{r}$$

Answer Sheets

Following are answer sheets for some of the graphical portions of the test.

The Cylinder's Frame



The Rod's Frame