AAPT UNITED STATES PHYSICS TEAM
AIP 2020

2020 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet. The only scratch paper you may use is scratch paper provided by the proctor. You may not use your own.
- This test contains 25 multiple choice questions. Select the answer that provides the best response to each question. Please be sure to use a No. 2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily of the same level of difficulty.
- Correct answers will be awarded one point; incorrect answers or leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones cannot be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- The question booklet, the answer sheet, and all scratch paper will be collected at the end of this exam.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 1, 2020.**

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN!

We acknowledge the following people for their contributions to this year's exams (in alphabetical order):

Ariel Amir, JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Daniel Longenecker, Kye Shi, Brian Skinner, Paul Stanley, Mike Winer, and Kevin Zhou.

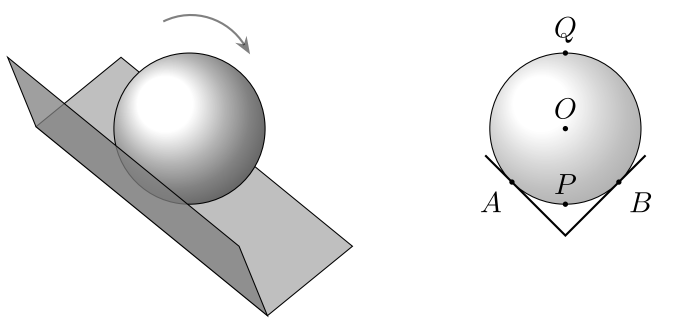
1. A ball is launched straight toward the ground from height h. When it bounces off the ground, it loses half of its kinetic energy. It reaches a maximum height of $2h$ before falling back to the ground again. What was the initial speed of the ball?

   (A) $\sqrt{gh}$
   (B) $\sqrt{2gh}$
   (C) $\sqrt{3gh}$
   (D) $\sqrt{4gh}$
   (E) $\sqrt{6gh}$ ← **CORRECT**

Solution

The calculation is easiest in the framework of energy: $E_0 = mgh + \frac{1}{2}mv_0^2 = 2E_f = 2mg(2h)$. So solving for initial velocity we get $v_0 = \sqrt{6gh}$.

2. A rigid ball of radius R is rolling without slipping along the rib of a right-angle chute, as shown in the figure. Which of the point(s) of the ball have the maximum speed?



side (3d) view   cross-section (2d) view

   (A) All points have the same speed.
   (B) The contact points A and B.
   (C) The center O.
   (D) The lowest point P.
   (E) The highest point Q. ← **CORRECT**

Solution

The top point. A and B must be stationary because they are in contact with the rib and the ball is rolling without slipping. Because the velocity at any point is given by $\mathbf{v} + \omega \times \mathbf{r}$, if the part of the ball below the center has speed 0, the highest point of the ball must have the speed. Explicitly, one can show that

$$v_{\text{top}} = v\frac{2+\sqrt{3}}{\sqrt{3}}.$$

3. When an axe is swung with kinetic energy E directly at a piece of wood, the edge of the axe is buried a depth L into the wood. If the axe is swung with kinetic energy $2E$, how deep will it be buried into the wood? Assume that the axe is wedge-shaped with a constant angle and that the force per unit contact area between the axe and the wood during the impact is proportional to the depth.

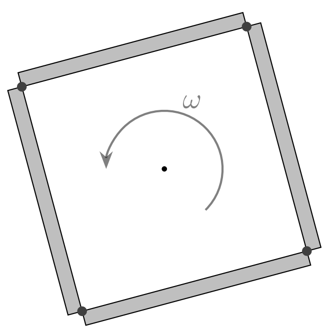
   (A) $2^{1/4}L$
   (B) $2^{1/3}L$ ← **CORRECT**
   (C) $\sqrt{2}L$
   (D) $2L$
   (E) $4L$

Solution

The energy required to enter the wood scales as L^3. This is because the energy to displace a harmonic oscillator scales as L^2, and the amount of contact with the wood gives another factor of d. Thus, doubling the energy will increase the distance into the wood by a factor of $2^{1/3}$.

Calculus is not necessary here, but may aid understanding this solution. We can define a spring constant density $\frac{k}{\ell}$ so that the energy per unit length is $\frac{ky^2}{2\ell}$, where $y = rL$ is the perpendicular displacement, and r is the wedge ratio. We can then integrate the energy as a function of distance into the wood: $\int_0^L dL' \frac{kL'^2}{2r^2\ell} = \frac{k}{6r^2\ell}L^3$.

4. Four identical rods, each of mass m and length $2d$, are joined together to form a square. The square is then spun around its center, as shown in the figure, at an angular frequency of ω. What is the magnitude of the force that the joints between the rods (at the corners of the square) must bear?



   (A) $m\omega^2 d/2$
   (B) $m\omega^2 d/\sqrt{2}$ ← **CORRECT**
   (C) $m\omega^2 d$
   (D) $\sqrt{2}m\omega^2 d$
   (E) $2m\omega^2 d$

Solution

Consider the rightmost rod, and consider an arbitrary point (d, h) on this rod. The centripetal acceleration at that point is given by $\omega^2 \mathbf{r} = \omega^2(d, h)$.

Thus, the rightward component of centripetal acceleration is $\omega^2 d$, which is constant along the rod. Thus, the rightward force on the rod is $F = ma = m\omega^2 d$.

Using Newton's third law and symmetry, we see that the tension in each joint must be in the direction perpendicular to the corresponding diagonal. Thus, we can set up a free-body diagram equation for each rod, with the two forces at the joints and the centripetal force.

Since these forces must cancel, we get $2F_{\text{joint}} \cos(\pi/4) = F_{\text{cent}} = m\omega^2 d$, so $F_{\text{joint}} = m\omega^2 d/\sqrt{2}$

5. A pendulum of length L oscillates inside a box. A person picks up the box and gently shakes it horizontally with frequency ω and a fixed amplitude for a fixed time. The final amplitude can be maximized if ω satisfies

   (A) $\omega = \sqrt{g/L}$ ← **CORRECT**
   (B) $\omega = 2\sqrt{g/L}$
   (C) $\omega = (1/2)\sqrt{g/L}$
   (D) There will be no effect on the amplitude for any value of ω.
   (E) None of the above

Solution

In the frame of the box, this is equivalent to a horizontal driving force with frequency ω. To increase the amplitude, the driving should be at resonance, so $\omega = \sqrt{g/L}$.

6. A planet is orbiting a star in a circular orbit of radius r_0. Over a very long period of time, much greater than the period of the orbit, the star slowly and steadily loses 1% of its mass. Throughout the process, the planet's orbit remains approximately circular. The final orbit radius is closest to

   (A) $1.02r_0$
   (B) $1.01r_0$ ← **CORRECT**
   (C) r_0
   (D) $0.99r_0$
   (E) $0.98r_0$

Solution

The angular momentum $L = mvr$ is conserved. Moreover, since the orbit is always circular, the kinetic energy and potential energy are related by $2K = -U$, giving $mv^2 = GMm/r$, where M is the mass of the star. Therefore $M \propto v^2 r \propto L^2/r$, so a 1% decrease in the star's mass gives a 1% increase in the orbital radius.

7. An astronaut standing on the exterior of the international space station wants to dispose of three pieces of trash. They face the station's direction of travel with the Earth to their left. From the astronaut's perspective, the three pieces are thrown (I) left, (II) right, and (III) up. To the astronaut's frustration, some of the pieces of trash return to the space station after several hours. They are

(A) II only

(B) III only

(C) I and II

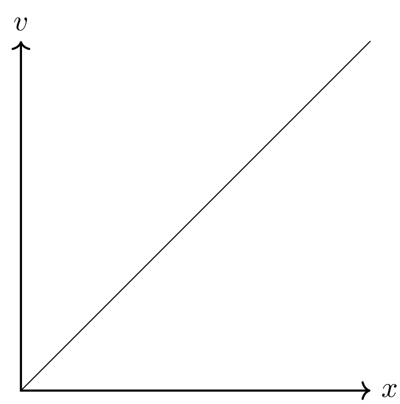
(D) II and III

(E) I, II, and III ← **CORRECT**

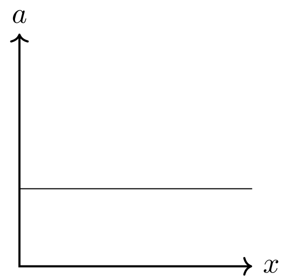
Solution

By Kepler's third law, $T^2 \propto a^3$ where a is the semimajor axis, and $E = -GMm/2a$. The astronaut's throw has a negligible impact on the total energy of the trash, and hence a negligible impact on a and hence T. In fact, after one full orbit, all three pieces of trash simply return to the space station.

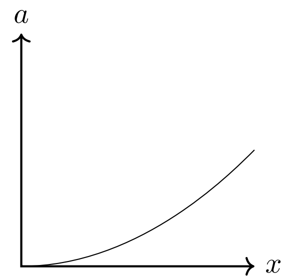
8. The velocity versus position plot of a particle is shown below. Which following choices is the correct acceleration vs. position plot of the particle?



(A)



(D)



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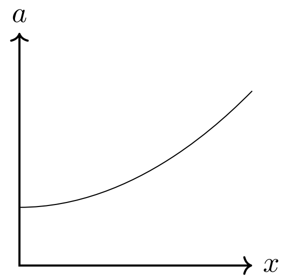
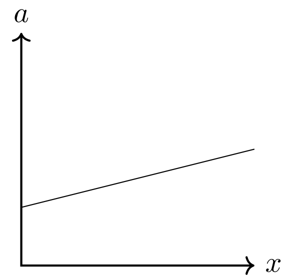
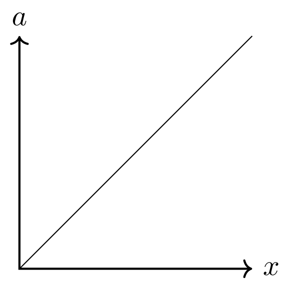


(E)



(C)



← **CORRECT**

Solution

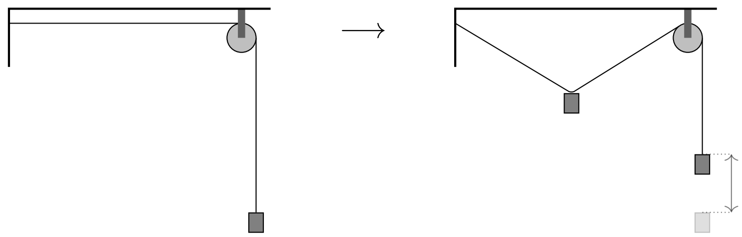
Suppose the particle starts at $x = 0$. Then its velocity is zero, meaning it remains at 0, and its acceleration is zero. So the correct acceleration plot must start at (0,0). It remains to determine whether the plot of acceleration vs time curves upward or is a straight line.

Suppose it is a straight line, as shown in choice (c). Then the average force on the particle traveling from some very small value to some x is proportional to x. The energy of the particle, which is the average force multiplied by x, is then proportional to x^2. Because the energy is the square of the velocity, we have $v^2 \propto x^2$, or simply $v \propto x$, as desired, so choice (c) is the one consistent with the shown v vs x plot.

For a calculus-based solution, the velocity is given by $v = kx$ for some k. Differentiating both sides with respect to time, we find $a = kv = k^2x$, so again we find that the acceleration is a straight line starting at zero and sloping upward.

The following information is relevant to problems 9 and 10.

9. A block of mass m is attached to a massless string. The string is passed over a massless pulley and the end of the string is fixed in place. The horizontal part of the string has length L. Now a small mass m is hung from the horizontal part of the string, and the system comes to equilibrium. (Diagram not necessarily to scale.)



Neglecting friction everywhere, the tension at the end of the string is

(A) $mg/2$

(B) mg ← **CORRECT**

(C) $3mg/2$

(D) $2mg$

(E) $3mg$

Solution

In equilibrium, the tensions on both sides of any pulley must be equal, for torque balance on the pulley. The same logic applies to the mass; one can think of it as just another pulley. Hence the tension everywhere in the string is mg.

10. During this process, the block has been raised by approximately a height

(A) $0.15L$ ← **CORRECT**

(B) $0.23L$

(C) $0.31L$

(D) $0.37L$

(E) $0.40L$

Solution

For force balance on the disc, the strings must attach to it at an angle of 30° to the horizontal; this also implies the mass must be placed at the middle of the horizontal section. Hence the horizontal section of string, originally of length L, becomes two tilted sections, each with length $L/\sqrt{3}$. The change in length is $L(2/\sqrt{3} - 1)$, which is equal to the increase in height of the block.

11. The maximal tension per area a material can sustain without failure is called its *tensile strength*. Plain steel has a tensile strength of 415 MPa. What is the maximal mass one can hang on a vertical steel rod of negligible mass and a diameter of 2 cm?

(A) 1300 kg

(B) 5200 kg

(C) 13 000 kg ← **CORRECT**

(D) 52 000 kg

(E) The answer depends on the length of the steel rod.

Solution

Balancing the forces, we have:

$$m_{\text{max}} g = \sigma \pi r^2$$

$$\to m_{\text{max}} = 13000\text{kg}$$

12. A point mass m is glued inside a massless hollow rod of length L at an unknown location. When the rod is pivoted at one end, the period of small oscillations is T. When the rod is pivoted at the other end, the period of small oscillations is $2T$. How far is the mass from the center?

    (A) $L/8$
    (B) $L/6$
    (C) $L/4$
    (D) $3L/10$ ← **CORRECT**
    (E) $2L/5$

Solution

We have $T \propto \sqrt{\Delta L}$, where ΔL is the distance from the pivot point to the mass. Then the ratio of distances is 4, so the mass is $0.2L$ from one of the ends, or $0.3L$ from the center.

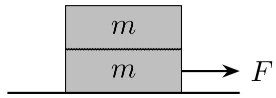
13. A ballerina with moment of inertia I is quickly twirling with angular velocity ω. In her hand she has a pen of mass m at a radius R from her axis of rotation. The ballerina releases the pen. Afterward, what happens to the vertical component of the angular momentum of the system consisting of the ballerina and the pen? You may ignore all friction, but not gravity or normal forces.

    (A) It decreases until the pen hits the floor.
    (B) It increases until the pen hits the floor.
    (C) It always stays the same. ← **CORRECT**
    (D) It initially stays the same, but decreases when the pen hits the floor.
    (E) It initially stays the same, but increases when the pen hits the floor.

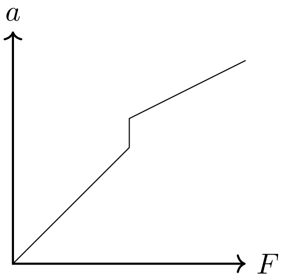
Solution

There are no external vertical torques in this problem, as both the gravity and normal forces are vertical, so the vertical component of angular momentum is conserved.

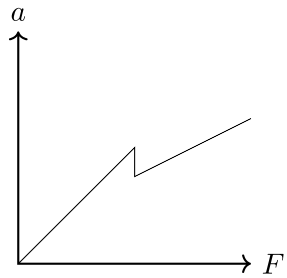
14. Two blocks of mass m are placed on top of each other, and the bottom block is placed on the ground. The ground is frictionless. The static and kinetic coefficients of friction between the two blocks are μ_s and μ_k, with $\mu_s < \mu_k$. The blocks are at rest initially. When a constant horizontal force F is then applied to the bottom block, which of the following graphs could show its acceleration as a function of F?



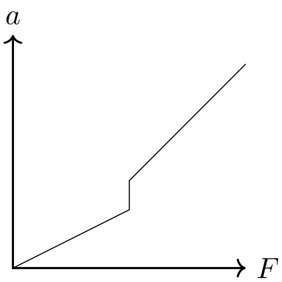
(A)



(D)

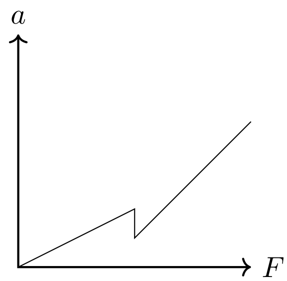


(B)

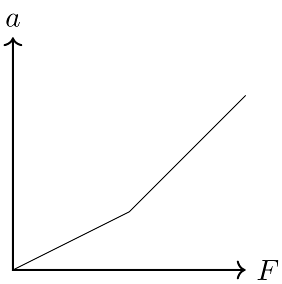


← **CORRECT**

(E)



(C)

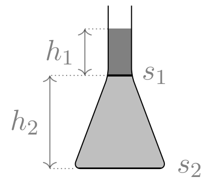


Solution

The statement $\mu_s < \mu_k$ above is a typo, which makes the setup not make physical sense. As a result, credit was given for all answers. The intended solution (for $\mu_k < \mu_s$) is as follows.

For small forces, the blocks move together, and the acceleration is linear in the force. At a certain threshold, the top block begins to slip. Because $\mu_k < \mu_s$, the friction force decreases suddenly at this threshold, so the acceleration of the bottom block increases suddenly. For higher forces, the acceleration is again linear in the force, but with a greater slope, since the top block is not moving with the bottom block.

15. As shown in the figure, a vessel contains two types of liquid: the liquid with density ρ_1 on top and ρ_2 on the bottom. The depth of the top liquid is h_1, and the interface area between the top and the bottom liquid is s_1. The bottom liquid has a depth of h_2. The area of the bottom of the vessel is s_2. What is the gauge pressure (i.e. pressure in excess of atmospheric pressure) at the bottom of the vessel?



(A) $(\rho_1 h_1 + \rho_2 h_2)g$ ← **CORRECT**

(B) $\dfrac{\rho_1 s_1 h_1 + \rho_2 s_2 h_2}{s_2} g$

(C) $\dfrac{1}{2}(\rho_1 + \rho_2)(h_1 + h_2)g$

(D) $(\rho_1 + \rho_2)(h_1 + h_2)g$

(E) $\rho_2(h_1 + h_2)g$

Solution

The gauge pressure is zero by definition at the top, so it is $\rho_1 h_1 g$ at the liquid interface. The pressure then increases by $\rho_2 h_2 g$ from here to the bottom of the vessel, giving the answer, $(\rho_1 h_1 + \rho_2 h_2)g$.

16. Liquid droplets store a given amount of potential energy per unit surface area, due to their surface tension. When two identical, nearly spherical liquid droplets coalesce on a certain type of surface, part of this energy can be converted into upward kinetic energy, causing the coalesced droplet to jump. Assuming the conversion is 100% efficient, how does the maximum height h depend on the radius r of the initial droplets?

(A) $h \propto r$

(B) $h \propto r^{1/2}$

(C) $h \propto r^{-1/2}$

(D) $h \propto r^{-1}$ ← **CORRECT**

(E) $h \propto r^{-2}$

Solution

The change in surface area, and hence the energy released, is proportional to r^2. The mass is proportional to r^3. Since the energy released is equal to mgh, we have $h \propto r^{-1}$.

17. Paul the Giant stands outside on a force-meter calibrated in Newtons, which reads 5000 N. Paul is wearing a large cowboy hat, which has horizontal cross-sectional area $A = 1\,\mathrm{m}^2$ and completely covers both him and the scale when seen from directly above. At time $t = 0$, rain begins to fall vertically downward on Paul, and any rain that hits his hat is collected in the hat's brim. The raindrops have a constant downward speed of 1 m/s, and the rain accumulates on the ground at a rate of 1 mm/s. What is the reading (in N) on the scale as a function of the time $t > 0$ (in s)? The density of water is $1000\,\mathrm{kg/m^3}$.

(A) $5001 + 11t$

(B) $5001 + 10t$ ← **CORRECT**

(C) $5000 + 11t$

(D) $5001 + 1.1t$

(E) $5001 + t$

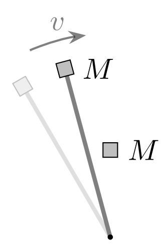
Solution

The force on the scale has three components: (1) Shrek's weight, (2) the weight of the water collected in the ogre hat, and (3) the pressure associated with changing the momentum of the downward-falling rain. Component (1) gives a constant weight 5000 N. Component (2) is equal to the weight of the water collected in the hat. The total volume of water collected is $V = (1\,\text{mm/s}) \times A \times t$, and the weight of the water is $V \times (1000\,\text{kg/m}^3) \times g = 10 \times t$.

Component (3) can be found by noting that in a time t, the total mass $M = \rho V$ of water that hits the hat has its momentum changed from $p = Mv$ to zero. Here $\rho = 1000\,\text{kg/m}^3$ is the density of water. The force applied by the hat on the water is therefore $F = \Delta p/\Delta t = Mv/t = 1$ N.

Adding these three contributions together gives a total weight $W = 5000 + 1 + 10t$.

18. A massless rigid rod is pivoted at one end, and a mass M is at the other end. Originally, the rod rotates frictionlessly about the pivot with a uniform angular velocity such that the mass M has speed v. The rotating rod collides with another mass M at its midpoint, which then sticks to the rod. After the collision, what is the kinetic energy of the system?



(A) $\frac{1}{4}Mv^2$

(B) $\frac{1}{3}Mv^2$

(C) $\frac{7}{18}Mv^2$

(D) $\frac{2}{5}Mv^2$ ← **CORRECT**

(E) $\frac{1}{2}Mv^2$

Solution

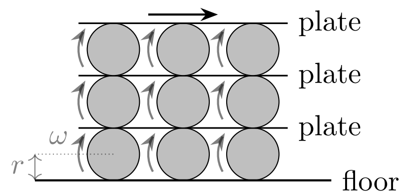
Angular momentum about the pivot is conserved in the collision. Let the final speed of the original mass M be v'. Then the final speed of the second mass is $v'/2$. Balancing the initial and final angular momentum,

$$Mvr = Mv'r + M(v'/2)(r/2)$$

which gives $v' = (4/5)v$. The final kinetic energy is

$$\frac{1}{2}M(v')^2 + \frac{1}{2}M(v'/2)^2 = \frac{2}{5}Mv^2.$$

19. A system of cylinders and plates is set up as shown. The cylinders all have radius r, and roll without slipping to the right with angular velocity ω. What is the speed of the top plate?



(A) ωr

(B) $2\omega r$

(C) $3\omega r$

(D) $4\omega r$

(E) $6\omega r$ ← **CORRECT**

Solution

Instantaneously, each cylinder can be thought of as rotating about its point of contact with the ground or plate below. The rotational angular velocity is ω. This means that, relative to the bottom of the cylinder, the top is moving at speed $2\omega r$. So the first plate is moving to the right at $2\omega r$. The second plate moves to the right at $2\omega r$ faster than the first, and the top plate moves to the right at $6\omega r$.

20. A car is driving against the wind at a constant speed v_0 relative to the ground. The wind direction is always opposite to the car's velocity, but its speed fluctuates about an average speed of v relative to the ground. The air drag force is Av_{rel}^2, where A is a constant and v_{rel} is the relative speed between the car and the wind. What is the average rate $\overline{P}$ of energy dissipation due to the air resistance?

(A) $\overline{P} = Av_0(v_0 + v)^2$

(B) $\overline{P} > Av_0(v_0 + v)^2$ ← **CORRECT**

(C) $\overline{P} < Av_0(v_0 + v)^2$

(D) Both (B) and (C) are possible depending on how v fluctuates.

(E) Both (A) and (C) are possible depending on how v fluctuates.

Solution

Define $u = v_0 + v$. Then the relative speed of the car and wind is $u + \delta u$, where δu averages to zero. The instantaneous energy dissipation rate is

$$P = Fv_0 = Av_0(u + \delta u)^2 = Av_0(u^2 + 2u\,\delta u + (\delta u)^2).$$

On average, the second term in parentheses will be zero, because δu averages to zero. But the third term is positive. So if the wind fluctuates at all, then on average,

$$\overline{P} > Av_0 u^2 = Av_0(v_0 + v)^2.$$

21. A circular table has radius R and $N > 2$ equally spaced legs of length h attached to its perimeter. Suppose the table has a uniform mass density with total mass m, and neglect the mass of the legs. Assuming the table does not slip, the minimum horizontal force needed to tip over the table is

(A) $\dfrac{mgR}{h}$

(B) $\dfrac{mgR}{h} \sin\left(\dfrac{N-2}{2N}\pi\right)$ ← **CORRECT**

(C) $\dfrac{mgR}{h} \cos\left(\dfrac{\pi}{N}\right)$ ← **CORRECT**

(D) $\dfrac{mgR}{h} \tan\left(\dfrac{N-2}{2N}\pi\right)$

(E) $\dfrac{mgR}{h} \sin\left(\dfrac{\pi}{2N}\right)$

Solution

Label the tops of two adjacent legs by A and B. The best way to push is to push directly opposite the table from the midpoint of AB, directly towards this midpoint.

When the table is about to tip over, these two legs will bear all of the weight of the table, and hence provide all of the frictional force. Balancing torques on the table about the axis AB, we have $Fh = mgR'$ where $R' = R\cos(\pi/N)$ is the distance from the center ofthe table to the midpoint of AB. Solving for F gives the answer. Note that credit was given for both choices (B) and (C), since they are equivalent.

22. A collision occurs between two masses. In each inertial reference frame, one can compute the change in total momentum $\Delta\mathbf{P}$ and the change in total kinetic energy ΔK due to the collision. Which of the following is true?

(A) $\Delta\mathbf{P}$ and ΔK do not depend on the frame. ← **CORRECT**

(B) $\Delta\mathbf{P}$ and ΔK do not depend on the frame for perfectly elastic collisions, but $\Delta\mathbf{P}$ may depend on the frame for inelastic collisions.

(C) $\Delta\mathbf{P}$ and ΔK do not depend on the frame for perfectly elastic collisions, but ΔK may depend on the frame for inelastic collisions.

(D) $\Delta\mathbf{P}$ and ΔK do not depend on the frame for perfectly elastic collisions, but both may depend on the frame for inelastic collisions.

(E) $\Delta\mathbf{P}$ and ΔK may both depend on the frame, for both perfectly elastic and inelastic collisions.

Solution

In a collision, momentum is conserved whether the collision is elastic or inelastic. This means $\Delta\mathbf{P} = 0$ regardless of reference frame, so $\Delta\mathbf{P}$ does not depend on frame.

In an elastic collision, $\Delta K = 0$, which is again independent of frame. In an inelastic collision, whatever kinetic energy is dissipated will go to heat up the masses (assuming they are isolated, and energy can only go into thermal energy). The heat capacity and temperature change of the masses are both frame-invariant, so the kinetic energy dissipated is again independent of reference frame. (This can also be shown more straightforwardly by direct calculation.)

23. Steve determines the spring constant k of a spring by applying a force F to it and measuring the change in length Δx. The tools he uses to measure F and Δx both have a constant absolute uncertainty, leading to an uncertainty in k of δk_S. If Tiffany measures the same spring constant with the same tools but by using a force that is five times larger, what will her uncertainty in k be in terms of δk_S?

(A) $\delta k_T = 0.04\,\delta k_S$

(B) $\delta k_T = 0.08\,\delta k_S$

(C) $\delta k_T = 0.2\,\delta k_S$ ← **CORRECT**

(D) $\delta k_T = 0.4\,\delta k_S$

(E) $\delta k_T = 0.5\,\delta k_S$

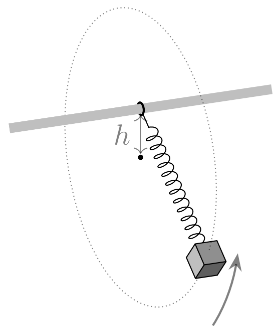
Solution

Conceptually, quintupling all forces, and consequently Δx (including the errors) would leave the error in k the same. So quintupling the values but not the errors multiplies the error in k by 0.2. Quantitatively, we use the equation for relative errors:

$$\left(\frac{\delta k}{k}\right)^2 = \left(\frac{\delta F}{F}\right)^2 + \left(\frac{\delta x}{\Delta x}\right)^2$$

The denominators on the right hand side will quintuple when Tiffany does the measurement, hence, the new uncertainty is $\frac{1}{5}\delta k_S$.

24. A mass m is connected to one end of a zero-length spring with spring constant k. The other end of the spring is connected to a frictionless bearing mounted around a horizontal pole so that the mass can swing in a vertical circle of radius R around the pole. The setup is shown in the figure below. What is the vertical distance h between the center of the circular orbit and the axis of the pole? Assume that both the diameter of the pole and the rest length of the spring are negligible compared to R.



(A) $\sqrt{mgR/k}$

(B) $R\sqrt{(R + mg/k)/(R - mg/k)}$

(C) $R - mg/k$

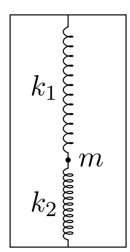
(D) mg/k ← **CORRECT**

(E) $\sqrt{R^2 - (mg/k)^2}$

Solution

Consider the instant when the mass is moving vertically upward. In this instant the mass's acceleration is perfectly horizontal, which means that the vertical component of the force from the spring must be equal and opposite to the force of gravity. The angle θ that the spring makes with the horizontal satisfies $\tan\theta = h/R$, where h is the desired vertical distance. The length of the spring in this instant is $\sqrt{h^2 + R^2}$, so the upward force is $k\sqrt{h^2 + R^2}\sin\theta = kh$. This should be set equal to mg, which means $h = mg/k$.

25. A ball of negligible radius and mass m is connected to two ideal springs. Each spring has rest length ℓ_0. The springs are connected to the ball inside a box of height $2\ell_0$, and the ball is allowed to come to equilibrium, as shown. Under what condition is this equilibrium point stable with respect to small horizontal displacements?



(A) $k_1 > k_2$

(B) $k_2 > k_1$

(C) $k_1 - k_2 > mg/\ell_0$ ← **CORRECT**

(D) $k_1 k_2/(k_1 + k_2) > mg/\ell_0$

(E) $k_1 k_2/(k_1 - k_2) > mg/\ell_0$

Solution

Suppose we call z the height measured from the middle of the box and x the horizontal displacement, also measured from the middle of the box. When the ball is at equilibrium at some z_0, the springs obey

$$-k_1 z_0 - k_2 z_0 = mg.$$

We can use this to find the equilibrium position,

$$z_0 = \frac{-mg}{k_1 + k_2}.$$

Imagine a horizontal displacement of size x. If x is small, the tension in the top spring is not changed from the tension at equilibrium (to first order in x). So the restoring force is

$$F_1 = T_1 \sin\theta \approx T_1 \theta \approx \frac{-k_1 x z_0}{\ell_0 - z_0}$$

where $\theta \approx \frac{x}{\ell_0 - z_0}$ is the angle the spring makes with the vertical.

The bottom spring similarly exerts a force

$$F_2 \approx \frac{k_2 x z_0}{\ell_0 + z_0}$$

pushing the ball away from equilibrium.

For equilibrium to be stable, for positive x, we must have

$$F_1 + F_2 < 0$$

Putting in F_1 and F_2, we have

$$-\frac{k_1 x z_0}{\ell_0 - z_0} + \frac{k_2 x z_0}{\ell_0 + z_0} < 0$$

Dividing by x and z_0, which we recall is negative,

$$\frac{k_1}{\ell_0 - z_0} < \frac{k_2}{\ell_0 + z_0}.$$

Plugging in our expression for z_0 and rearranging this, we derive

$$k_1 - k_2 < \frac{mg}{\ell_0}.$$