AAPT UNITED STATES PHYSICS TEAM
AIP 2021

2021 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout this contest.
- You may write in this booklet of questions. However, you will not receive any credit for anything written in this booklet. You may only use the scratch paper provided by the proctor.
- This test contains 25 multiple choice questions. Select the answer that provides the best response to each question. Please be sure to use a No.2 pencil and completely fill the box corresponding to your choice. If you change an answer, the previous mark must be completely erased. Only the boxes preceded by numbers 1 through 25 are to be used on the answer sheet.
- All questions are equally weighted, but are not necessarily of the same level of difficulty.
- Correct answers will be awarded one point; incorrect answers or leaving an answer blank will be awarded zero points. There is no additional penalty for incorrect answers.
- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- The question booklet, the answer sheet and the scratch paper will be collected at the end of this exam.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers or solutions) on this contest until after February 1, 2021.**

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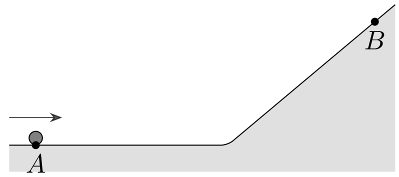
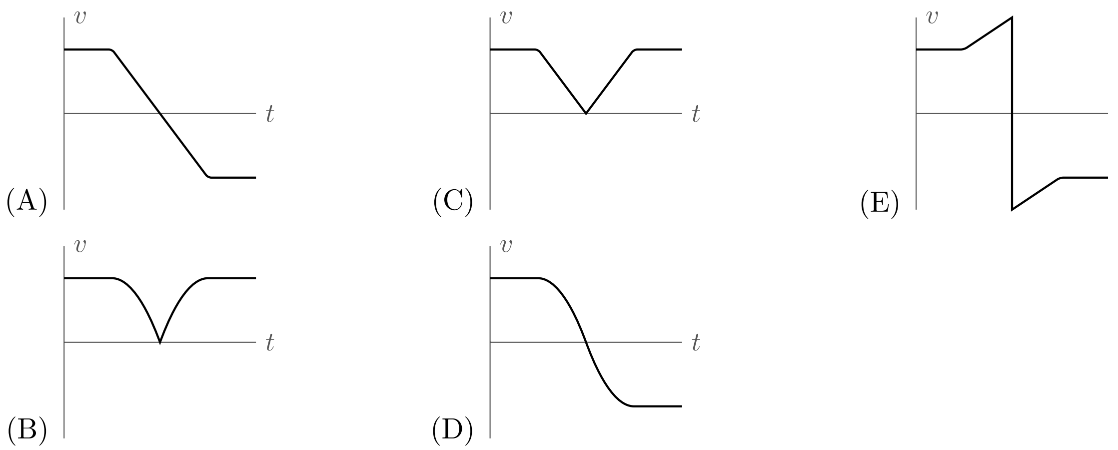
We acknowledge the following people for their contributions to this year's exams (in alphabetical order):

JiaJia Dong, Mark Eichenlaub, Abi Krishnan, Kye Shi, Brian Skinner, Mike Winer and Kevin Zhou

The following information applies to problems 1 and 2.
At time $t = 0$, a small ball is released on the track shown, with an initial rightward velocity. Assume the ball always rolls along the track without slipping.



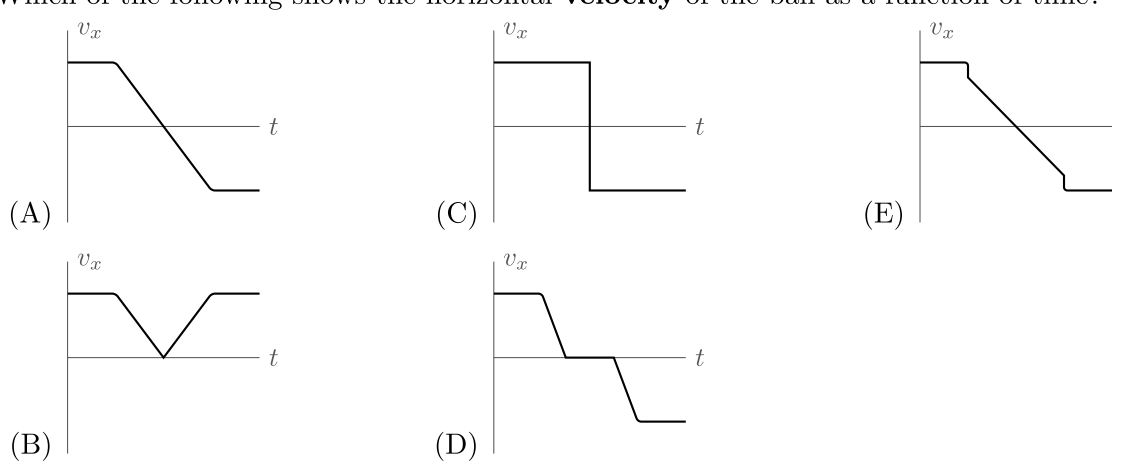
1. The ball starts at point A, turns around at point B, and returns to point A. Which of the following shows the **speed** of the ball as a function of time?



Solution

The speed starts uniform, then uniform decelerates to zero as the ball rises up the slope, and is instantaneously zero when the ball is at point B. Then the ball speeds up again as it goes down the slope, and returns to point A with the same speed it started with.

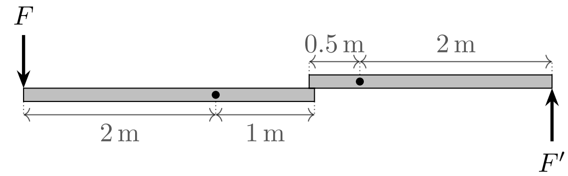
2. Which of the following shows the horizontal **velocity** of the ball as a function of time?



Solution

The horizontal velocity is initially constant. As in the previous question, when the ball goes through the bend in the track, its speed is unchanged. However, the direction of its velocity quickly changes, so that only part of it is directed horizontally. Thus, the horizontal velocity very quickly drops. As the ball climbs up the hill, its horizontal velocity decreases uniformly, then very quickly increases in magnitude when it goes through the curve again. Therefore, the answer is (E).

3. Two massless rods are attached to frictionless pivots, with their ends touching. The distances between the pivot points and the endpoints of the rods are shown below.



Neglecting friction between the rods, if a force F is applied at the left end of the left rod, what force F' must be applied at the right end of the right rod to keep the system in equilibrium?

(A) $F/8$
(B) $F/2$ ← **CORRECT**
(C) $4F/7$
(D) $6F/5$
(E) $2F$

Solution

By balancing torques on the first rod, we see that the normal force on the right end is twice the force exerted on the left end; that is, the first rod is acting as a lever with mechanical advantage 2. By similar logic, the second rod acts as a lever with mechanical advantage 1/4. Thus, the combination of the two has mechanical advantage $2(1/4) = 1/2$, so $F' = F/2$.

4. Alice and Bethany stand side by side on the Earth's equator. If Alice jumps directly upward, in her frame of reference, to a small height h much less than the radius of the Earth, she will land a distance D to the west of Bethany. If Alice had instead jumped to a height $2h$, how far to the west of Bethany would she land? Neglect air resistance.

(A) $D/\sqrt{2}$
(B) D
(C) $\sqrt{2}\,D$
(D) $2D$
(E) $2^{3/2}D$ ← **CORRECT**

Solution

The distance to the left of the starting position is given by $\Delta v\, t$, where $\Delta v = v_{\text{ave}} - v_0$ is the average change in tangential velocity due to the jump and t is the total time of the jump. We can compute the proportionality between Δv and h using conservation of angular momentum,

$$v_{\text{ave}}(R + h_{\text{ave}}) \approx v_0 R \implies v_{\text{ave}} \approx \frac{v_0 R}{R + h_{\text{ave}}}.$$

We use an $\approx$ because $[v \cdot (R+h)]_{\text{ave}} \neq v_{\text{ave}} \cdot (R + h_{\text{ave}})$, but they both exhibit the same scaling. Then,

$$\Delta v = v_0 - v_{\text{ave}} = \frac{v_0 h}{R + h_{\text{ave}}} \implies \Delta v \propto h.$$

From basic kinematics, $t \propto h^{1/2}$, so $\Delta v\, t \propto h^{3/2}$, so the answer is (E).

5. A train starts from city A and stops in city B. The distance between the cities is s. The train's maximal acceleration is a_1 and its maximal deceleration is a_2 (in absolute value). What is the shortest time in which the train can travel between A and B?

(A) $2\sqrt{\dfrac{s}{a_1+a_2}}$

(B) $2\sqrt{\dfrac{s}{\sqrt{a_1a_2}}}$

(C) $\sqrt{2\dfrac{s(a_1+a_2)}{a_1a_2}}$ ← **CORRECT**

(D) $\sqrt{\dfrac{2sa_2}{a_1(a_1+a_2)}}$

(E) $\sqrt{\dfrac{s\sqrt{a_1a_2}}{(a_1+a_2)^2}}$

Solution

When going from city to city as quickly as possible, the train accelerates for some time t_1, reaches a maximum speed v, then immediately decelerates for a time t_2, coming to rest right when it arrives at the new city. The speed v obeys $v = a_1t_1$ and $v = a_2t_2$, and so

$$a_1t_1 = a_2t_2 = v.$$

The average speed of the train is $v/2$, so we also have

$$(t_1 + t_2)\frac{v}{2} = s.$$

If we solve these equations for t_1, we find

$$t_1 = \sqrt{\frac{2a_2s}{a_1(a_1+a_2)}}.$$

Then we can find that

$$t_2 = \frac{a_1t_1}{a_2},$$

and therefore

$$t = t_1 + t_2 = t_1\left(\frac{a_1+a_2}{a_2}\right).$$

Substituting in our equation for t_1, we have

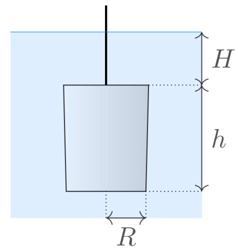
$$t = \sqrt{\frac{2s(a_1+a_2)}{a_1a_2}}.$$

Alternatively, we can consider the limit $a_2 \to \infty$. In this case, the train accelerates all the way to the next city, and the time is

$$t_{a_2\to\infty} = \sqrt{\frac{2s}{a_1}}.$$

Only the correct answer simplifies appropriately.

6. A cylindrical bucket of negligible mass has radius R and height h, and is open at the top. It is submerged in water of density ρ, with its top a distance H below the surface. How much work is needed to pull the bucket slowly up so that its bottom is just above the lake surface?



(A) $\rho g \pi R^2 h(H - h)$
(B) $\rho g \pi R^2 h^2$
(C) $\rho g \pi R^2 (H - h)^2$
(D) $\rho g \pi R^2 h^2/2$ ← **CORRECT**
(E) $\rho g \pi R^2 h(H - h)/2$

Solution

While the bucket is in the lake, it is full of water, and the weight of the water is exactly cancelled by the buoyant force. Thus, no work is required to pull the bucket until its top is at the lake surface. From this point onward, the force linearly increases. The average force required to pull the bucket out of the water is $mg/2$ where $m = \pi \rho R^2 h$ is the mass of the water in the bucket, and the distance is h, giving total work $\rho g \pi R^2 h^2/2$.

The problem can also be solved using energy conservation. After the bucket has been lifted, a mass m of water has been raised from the water surface to height $h/2$ above it, giving work $mgh/2$ and the same result.

7. A point mass slides with speed v on a frictionless horizontal surface between two fixed parallel walls, initially a distance L apart. It bounces between the walls perfectly elastically. You move one of the walls towards the other by a distance $0.01L$, with speed $0.0001v$. What is the final speed of the point mass?

(A) $1.001v$
(B) $1.002v$
(C) $1.005v$
(D) $1.01v$ ← **CORRECT**
(E) $1.02v$

Solution

The total time is $100L/v$, during which 50 round-trips occur. Because the collisions are elastic, each collision with the moving wall increases the speed of the block by $0.0002v$. Thus, the final speed is increased by $(50)(0.0002v) = 0.01v$.

8. A mint produces 100,000 coins. Upon weighing some of them with a precise scale, officials find that the coins vary slightly in weight, with an independent uncertainty of 1%. About how many coins must be randomly sampled and weighed in order to determine the total weight of the coins to within 0.1% uncertainty? Assume there are no sources of systematic uncertainty.

(A) Almost all of the coins must be weighed.

(B) 10,000

(C) 1,000

(D) 100 ← **CORRECT**

(E) 10

Solution

Since the uncertainty comes from the independent 1% of each coin, to reduce the uncertainty by a factor of 10, one needs to take $10^2 = 100$ samples.

9. NASA trains astronauts to experience weightlessness with an airplane which flies in a parabolic arc with constant acceleration g toward the ground. The plane can remain on this trajectory for at most 25 seconds, due to the large change in altitude required. If instead of simulating weightlessness, NASA wanted to fly a trajectory that would simulate the gravitational acceleration of Mars 3.7 m/s^2, for what length of time can the plane simulate Mars gravity? Assume that the maximum change in altitude is the same for both trajectories.

(A) 25 s

(B) 31 s ← **CORRECT**

(C) 41 s

(D) 68 s

(E) 183 s

Solution

The new acceleration of the plane should be $10\text{ m/s}^2 - 3.7\text{ m/s}^2 = 6.3\text{ m/s}^2$. The altitude change is supposed to be the same, so if the weightless flight took time t and the Mars flight took time τ, we have

$$\frac{1}{2}10\text{ m/s}^2 \cdot t^2 = \frac{1}{2}6.3\text{ m/s}^2 \cdot \tau^2.$$

Solving for τ gives 31 s.

10. A uniform solid circular disk of mass m is on a flat, frictionless horizontal table. The center of mass of the disk is at rest and the disk is spinning with angular frequency $omega_0$. A stone, modeled as a point object also of mass m, is placed on the edge of the disk, with zero initial velocity relative to the table. A rim built into the disk constrains the stone to slide, with friction, along the disk's edge. After the stone stops sliding with respect to the disk, what is the angular frequency of rotation of the disk and stone together?

(A) ω_0

(B) $2\omega_0/3$

(C) $\omega_0/2$ ← **CORRECT**

(D) $\omega_0/3$

(E) $\omega_0/4$

Solution

The center of mass of the stone-disk system is initially stationary, so it remains stationary. This means the final motion of the system is purely rotation about its center of mass.

The disk and stone conserve angular momentum. If the angular momentum of the disk about its center is I_0, the initial angular momentum is

$$L_0 = I_0\omega_0.$$

We need to calculate the angular momentum of the stone-disk system about its center of mass.

The center of mass is half way between the center of the disk and the stone, so if the radius of the disk is R, and its mass m, the moment of inertia of the stone is

$$I_{\text{stone}} = m\left(\frac{R}{2}\right)^2.$$

By the parallel axis theorem, the angular momentum of the disk is

$$I_{\text{disk}} = m\left(\frac{R}{2}\right)^2 + I_0.$$

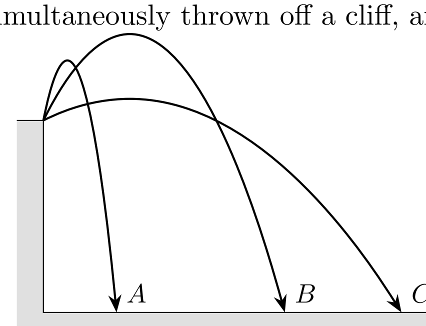
Using $I_0 = \frac{1}{2}mR^2$, the angular momentum of the stone-disk system is

$$I_{\text{total}} = mR^2 = 2I_0.$$

By angular momentum conservation, doubling the moment of inertia cuts the angular frequency in half, so the final angular frequency is

$$\omega = \frac{1}{2}\omega_0.$$

11. Projectiles A, B, and C are simultaneously thrown off a cliff, and take the trajectories shown.



Neglecting air resistance, rank the times t_A, t_B, and t_C they take to hit the ground.

(A) $t_A < t_B < t_C$

(B) $t_A < t_C < t_B$

(C) $t_C < t_B < t_A$

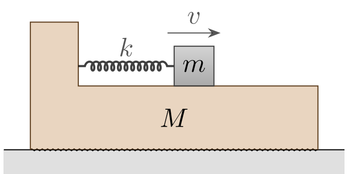
(D) $t_C < t_A < t_B$ ← **CORRECT**

(E) There is not enough information to decide.

Solution

Since the x and y directions are independent, the time to impact only depends on the initial v_y, which in turn determines the maximum height. The higher the maximum height, the longer the trajectory takes.

12. An object of mass $m = 1\,\text{kg}$ is attached to a platform of mass $M = 4\,\text{kg}$ with a spring of spring constant $k = 400\,\text{N/m}$. There is no friction between the object and the platform, and the coefficient of static friction between the platform and the ground is $\mu = 0.1$. The object is placed at its equilibrium position, and then given a horizontal velocity v.



For what v will the platform never slip on the ground?

(A) $v \leq 0.1\,\text{m/s}$

(B) $v \leq 0.2\,\text{m/s}$

(C) $v \leq 0.25\,\text{m/s}$ ← **CORRECT**

(D) $v \leq 0.4\,\text{m/s}$

(E) $v \leq 0.5\,\text{m/s}$

Solution

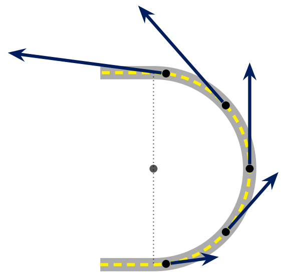
The force of the spring on the platform is $F_s = kx$, where x is the displacement of the object from equilibrium. The maximum value of x can be found by conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2$$

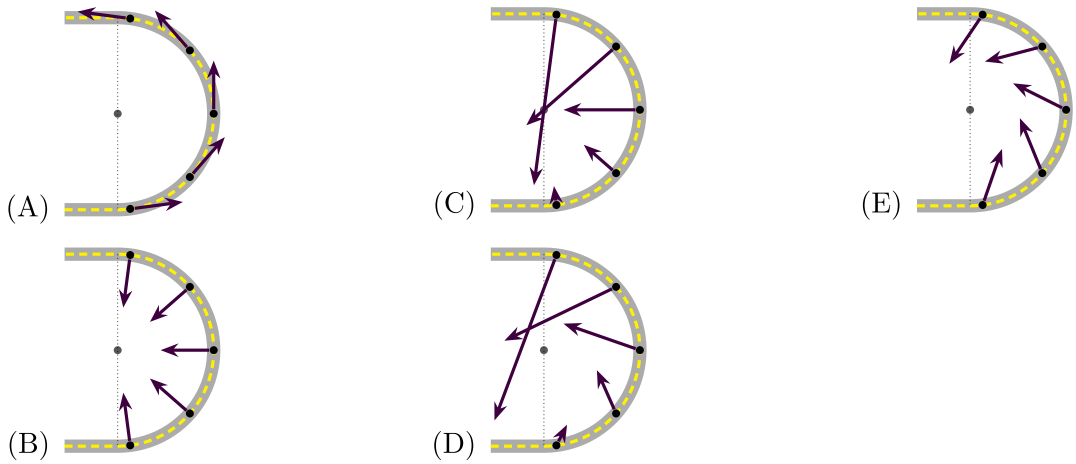
which gives a maximum spring force $F_s = v\sqrt{km}$ acting on the platform. The maximum possible friction force on the platform is $(M+m)\mu g$. Setting these two equal, the maximum velocity is

$$v = \frac{(M+m)\mu g}{\sqrt{km}} = 0.25\,\text{m/s}.$$

13. A car is driving on a semicircular racetrack. Its velocity at several points along the track is shown below.



Which of the following could depict its acceleration at the corresponding points?



Solution

Because the car is speeding up, its acceleration has a tangential component, i.e. a component parallel to the racetrack. Because the car is turning, its acceleration also has a component v^2/r directed towards the center of the track. Since v increases through the turn, this inward radial acceleration increases through the turn. The only choice with all of these features is (D).

14. A bacterium cell swims by rotating its bundle of flagella to counter a viscous drag force in the medium. The drag force $F(R, v, \eta)$ only depends on the typical length scale of the cell R, its speed v, and the viscosity of the fluid η, which has units of $\mathrm{kg/(m \cdot s)}$. It is observed under a microscope that a cell of length 1 μm swims at about 20 μm/s. Estimate the speed of a cell of length 0.5 μm, assuming cells of all sizes generate the same amount of force from their flagella.

(A) 5 μm/s

(B) 10 μm/s

(C) 40 μm/s ← **CORRECT**

(D) 80 μm/s

(E) There is not enough information to decide.

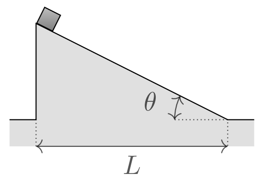
Solution

Using dimensional analysis, we can see that force scales as $F \sim Rv\eta$. To see this you can try:

$$[\mathrm{m}]^{\mathrm{a}}[\mathrm{m/s}]^{\mathrm{b}}[\mathrm{kg/(m \cdot s)}]^{\mathrm{c}} = [\mathrm{N}] = [\mathrm{kg \cdot m/s^2}].$$

The only solution is $a = b = c = 1$. Therefore, in the same medium, when the cell length reduces by 1/2, its speed doubles.

15. A block is released from rest at the top of a fixed, frictionless ramp with horizontal length L and inclination θ.



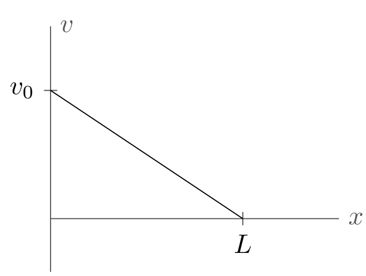
For a fixed value of L, which value of θ minimizes the time needed for the block to reach the bottom of the ramp?

(A) 30°
(B) 45° ← **CORRECT**
(C) 60°
(D) 75°
(E) 80°

Solution

The acceleration down the ramp is $a = g\sin\theta$, and $d = at^2/2$ where the distance $d = L/\cos\theta$. Hence $t^2 \propto d/a \propto 1/\sin\theta\cos\theta \propto 1/\sin(2\theta)$. The maximum of $\sin(2\theta)$, which minimizes t, is at $\theta = 45°$.

16. A particle begins at the point $x = 0$ and moves along the x-axis with initial rightward velocity v_0. Later, the particle reaches the point $x = L$. The velocity as a function of position during this time interval is shown below.



Consider the following three statements.

I. The particle instantaneously stops at $x = L$.
II. The particle is uniformly accelerated.
III. The particle could be performing simple harmonic motion.

Which of these statements are true?

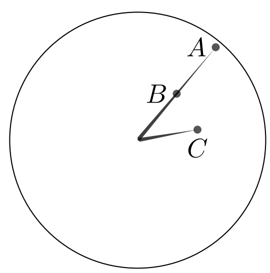
(A) Only I. ← **CORRECT**
(B) Only II.
(C) Both I and II.
(D) Both I and III.
(E) None of the above. ← **CORRECT**

Solution

Statements II and III are false. First, if the particle were decelerating uniformly, then by conservation of energy, $mv^2/2 + Fx$ would be constant for some constant F, which means $v(x)$ would not be a straight line. (Of course, $v(t)$ would be a straight line, but this isn't the same thing as $v(x)$.) Second, if the particle were performing simple harmonic motion, then $mv^2/2 + kx^2/2$ would be constant for some constant k, and the resulting $v(x)$ would also not be a straight line.

As for statement I, the particle would instantaneously stop at $x = L$, since $v(L) = 0$, but for the velocity profile shown, it only reaches the point $x = L$ asymptotically. Thus, either choice (A) or choice (E) could be construed as correct, and we accepted both answers.

17. In a certain country, the short hand of a clock is exactly half as long as the long hand, and rotates twice for each rotation of the long hand.



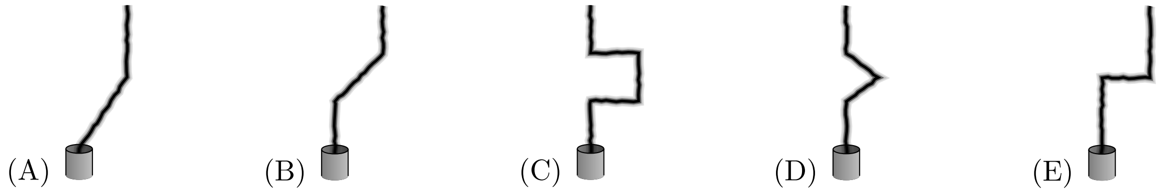
The three points shown on the clock hands have accelerations of magnitude a_A, a_B, and a_C. The point B is at the midpoint of the long hand. Which of the following is true?

(A) $a_A < a_B = a_C$
(B) $a_A = a_B < a_C$
(C) $a_B < a_A < a_C$ ← **CORRECT**
(D) $a_B < a_A = a_C$
(E) $a_B < a_C < a_A$

Solution

The centripetal acceleration is $a = \omega^2 r$. Since A and B have the same ω, $a_A = 2a_B$. Since B and C have the same r, $a_C = 4a_B$. Thus, $a_B < a_A < a_C$.

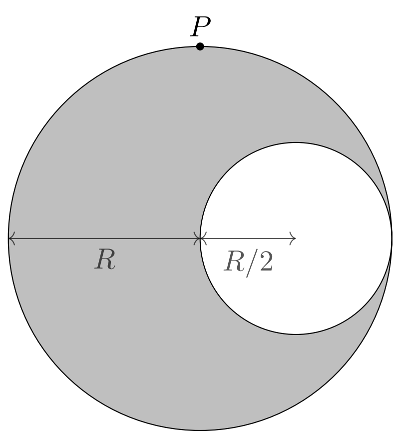
18. A factory's smokestack continually produces smoke. The smoke always rises with a constant speed relative to the air around it. Suppose that the air was initially still, then abruptly started blowing to the right, then abruptly became still again for some time. Which of the following could show the resulting shape of the smoke plume?



Solution

The smoke right next to the smokestack has just been emitted, and has only been exposed to still air, so it must rise straight upward. This rules out choice (A). The smoke high above the smokestack has experienced the full duration of the rightward wind, so it all must have been pulled to the right by the same amount, ruling out choices (C) and (D). Finally, the smoke in between has been exposed to only part of the rightward gust of wind (since some of that wind occurred before that smoke left the smokestack), so its path should smoothly connect the two vertical segments. This rules out choice (E), leaving (B) as the answer.

19. A cavity of radius $R/2$ is dug out of a spherical planet with uniform mass density of mass M and radius R. What is the magnitude of the gravitational field at point P in the diagram below?



(A) $(0.200)GM/R^2$
(B) $(0.457)GM/R^2$
(C) $(0.829)GM/R^2$
(D) $(0.900)GM/R^2$
(E) $(0.912)GM/R^2$ ← **CORRECT**

Solution

The gravitational field due to a completely filled gray sphere is $-\frac{MG}{R^2}\hat{y}$. We can treat the white circle as having negative mass. The center of the white circle is $\sqrt{5}R/2$ away from the point P. The field from the white circle has a magnitude of $\frac{MG}{8(R^2+(R/2)^2)} = \frac{MG}{10R^2}$. The field from the white circle is thus $\frac{MG}{5R^2\sqrt{5}}\hat{y} - \frac{MG}{10R^2\sqrt{5}}\hat{x}$. Adding the fields and taking the magnitude gives $(0.912)\frac{MG}{R^2}$.

20. The line between day and night on a planet or moon is called the *terminator*. How fast does the terminator of Earth's moon move across its surface at the equator? The radius of the moon is 1.74×10^6 m.

(A) 0 m/s
(B) 4.5 m/s ← **CORRECT**
(C) 83 m/s
(D) 465 m/s
(E) 2201 m/s

Solution

The same side of the moon always faces Earth. This means the terminator runs all the way around the moon once per month. The speed is

$$v = \frac{2\pi \cdot 1.74 \times 10^6 \text{ m}}{28 \text{ days}} \approx 4.5 \text{ m/s}.$$

21. A solid sphere sits at the top of a ramp of height h inclined at angle θ to the horizontal. Both the static and kinetic coefficients of friction between the sphere and incline are $\mu_k = \mu_s = 0.2$. The sphere is released from rest at the top of the incline. For which of the following values of θ is the total translational plus rotational kinetic energy of the sphere greatest when it reaches the bottom of the incline?

(A) 10° ← **CORRECT**
(B) 45°
(C) 60°
(D) 80°
(E) The mechanical energy is the same for all choices.

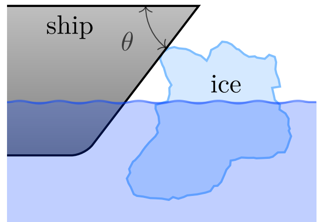
Solution

As long as the sphere rolls without slipping, it converts all its gravitational potential energy into kinetic energy, so its mechanical energy stays constant. That's what happens when the slope is very shallow. When the slope is steep, the acceleration is greater, and the torque required to roll without slipping is greater than what friction can provide. At these slopes, the sphere slips and dissipates mechanical energy, so its mechanical energy at the bottom of the slope is lower.

When rolling without slipping, a solid sphere has acceleration down an incline of $\frac{5}{7} g \sin\theta$.

The critical transition slope occurs when the friction force required to provide angular acceleration for the ball exceeds the maximum friction of $mg \cos\theta \mu$. This occurs when $\tan\theta = \frac{7}{2}\mu$, or about 35° when $\mu = 0.2$. This means that case (a) is rolling without slipping, while (b), (c), and (d) all have slipping and less kinetic energy at the bottom of the slope.

22. A ship rams into a hunk of ice floating in the sea. "Icebreakers" are ships designed so that when this happens, the ice is pushed down beneath the ship. If the coefficient of static friction between the ice and the ship is μ_s, what condition applies to the angle θ, as shown in the figure, so that the icebreaker functions as intended?



(A) $\cot\theta > \mu$ ← **CORRECT**

(B) $\cos\theta > \mu$

(C) $\cot\theta < \mu$

(D) $\cos\theta < \mu$

(E) It depends on the curvature of the ice.

Solution

There is a force from the ship on the ice, which may be broken into a normal force and a friction force. The vertical component of the normal force is $-N\cos\theta$ and the vertical component of the friction force is $f\sin\theta$ with f the magnitude. The friction force can be at most $N\mu$, so the vertical force from the ship on the ice is

$$F_y \leq \mu N\sin\theta - N\cos\theta.$$

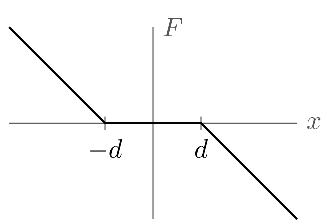
We want $F_y < 0$ in order to push the ice down, so we must have

$$N\cos\theta > \mu N\sin\theta$$

or

$$\cot\theta > \mu.$$

23. An imperfect spring has a restoring force F that depends on the displacement x from equilibrium as in the graph shown below.



The slope of the curve for $x < -d$ and $x > d$ is a constant $-k$. A mass m is attached to the spring and released from rest at the position $x = A$, where $A > d$. What is the period of the subsequent motion?

(A) $T = \sqrt{\frac{m}{k}}\left(2\pi + \frac{2d}{A}\right)$

(B) $T = \sqrt{\frac{m}{k}}\left(2\pi + \frac{2d}{A-d}\right)$

(C) $T = \sqrt{\frac{m}{k}}\left(2\pi + \frac{4d}{A-d}\right)$ ← **CORRECT**

(D) $T = \sqrt{\frac{m}{k}}\left(2\pi + \frac{\pi d}{A-d}\right)$

(E) $T = \sqrt{\frac{m}{k}}\left(2\pi + \frac{2\pi d}{A-d}\right)$

Solution

The full motion is the combination of two parts: simple harmonic motion as if the spring were ideal (when the mass is at $|x| > d$), and motion with uniform velocity in the region $|x| < d$. By conservation of energy, the velocity in this region satisfies

$$\frac{1}{2}mv^2 = \frac{1}{2}k(A-d)^2$$

so the time spent in this region is

$$\frac{4d}{v} = \frac{4d}{A-d}\sqrt{\frac{m}{k}}.$$

The time spent at $|x| > d$ is $2\pi\sqrt{m/k}$, and adding these two times gives the result.

24. Two satellites are in circular orbits around a star with equal radius r, speed v, and period T. The satellites are initially diametrically opposite each other. In order to meet the second satellite in time $T/2$, the first satellite should decrease its speed to approximately

(A) $0.50v$

(B) $0.64v$ ← **CORRECT**

(C) $0.71v$

(D) $0.76v$

(E) $0.82v$

Solution

For the meeting to occur, the first satellite must switch into an orbit with period $T/2$. By Kepler's third law, the new semi-major axis should be $a = r/2^{2/3}$. Since the total energy of a satellite is $E = -GMm/2a$, the energy difference of the two satellites is

$$E_2 - E_1 = \frac{GMm}{2r}(2^{2/3} - 1) = \frac{1}{2}mv^2 - \frac{1}{2}mv'^2$$

where v' is the speed of the first satellite right after it slows down. On the other hand, in circular motion we have $mv^2 = GMm/r$, which implies after some simplification that

$$v' = \sqrt{2 - 2^{2/3}}\,v = 0.64v.$$

25. Spaceman Fred's trusty pellet sprayer is held at rest a distance h away from the center of Planet Orb, which has radius $R \ll h$. The pellet sprayer ejects pellets radially outward, uniformly in the plane of the page. These pellets are all launched with the same speed v, so that a pellet launched directly away from Orb by the pellet sprayer can just barely escape it. What fraction of the pellets eventually lands on Orb? (Hint: you may use the small angle approximation, $\sin\theta \approx \theta$ for $\theta \ll 1$, where θ is in radians.)

Left: the pellet sprayer relative to Orb   Right: a close-up of the pellet sprayer

(A) $\frac{1}{2\pi}\frac{R}{h}$

(B) $\frac{1}{\pi}\frac{R}{h}$

(C) $\frac{2}{\pi}\frac{R}{h}$

(D) $\frac{1}{2\pi}\sqrt{\frac{R}{h}}$

(E) $\frac{1}{\pi}\sqrt{\frac{R}{h}}$ ← **CORRECT**

Solution

Because all pellets are launched at the escape velocity, the speed of one of these pellets a distance r away from Orb is

$$v(r) = \sqrt{\frac{2GM}{r}}.$$

At the distance r_{min} of closest approach, the pellet's velocity must be perpendicular to its displacement from the planet's center. Then, by conservation of angular momentum, for a pellet of mass m launched at an angle θ from the vertical,

$$mv(h)h\sin\theta = mv(r_{\text{min}})r_{\text{min}} \implies \sqrt{h}\sin\theta = \sqrt{r_{\text{min}}}.$$

The pellet collides with the planet whenever $r_{\text{min}} < R$. Then, our equation for θ_{max} is

$$\theta_{\text{max}} \approx \sin\theta_{\text{max}} = \sqrt{\frac{R}{h}}.$$

The range of collision angles is thus $-\theta_{\text{max}} < \theta < \theta_{\text{max}}$ and the range of allowed angles is 2π, so the fraction is

$$f = \frac{1}{\pi}\sqrt{\frac{R}{h}}.$$