AAPT UNITED STATES PHYSICS TEAM
AIP 2021

Traveling Team Selection Exam

Information About The 2021 USAPhO+

- The 2021 USAPhO+ is a 5-hour exam taking place on Saturday, May 8 from noon to 5 PM, Eastern time.
- The exam is hosted by AAPT on the platform provided by Art of Problem Solving. It will be proctored by the US Physics Team coaches via Zoom.
- Before you start the exam, make sure you are provided with blank paper, both for your answers and scratch work, writing utensils, **graph paper and a ruler**, a hand-held scientific calculator with memory and programs erased, and a computer for you to log into the USAPhO+ testing page.
- At the end of the exam, you have 20 minutes to upload solutions to all of the problems for that part. For each problem, scan or photograph each page of your solution, combine them into a single PDF file, and upload them on the testing platform.
- USAPhO+ graders are not responsible for missing pages or illegible handwriting. No late submissions will be accepted.

Congratulations again on your qualification for the USAPhO+. We wish you the best of luck on the challenging problems to follow.

We acknowledge the following people for their contributions to this year's exam (in alphabetical order):

JiaJia Dong, Mark Eichenlaub, Abijith Krishnan, Kye W. Shi, Brian Skinner, Mike Winer, and Kevin Zhou.

Question 1

The Jet Stream

The jet stream is an eastward wind current that moves over the continental United States at an altitude of $23,000$ to $35,000$ feet (the range of typical cruising altitudes of commercial airlines). This strong current affects flight times significantly: flights traveling eastward fly significantly faster than flights traveling westward.

This problem consists of two independent parts. In the first part, you will consider a simple model for airplane flight. In the second part, you will determine the jet stream speed on a fictitious planet called Orb.

1. The power that a plane expends is used both to combat drag and to generate lift. Throughout this part of the problem, you may assume that the plane travels with horizontal velocity $\mathbf{v}_{\text{rel}}$ relative to the air, the density of air is ρ_{air}, the mass of the plane is m, and the cross-sectional area of the plane is A_{cs}.

   (a) The drag force on an airplane is given by

   $$\mathbf{F}_{\text{drag}} = -\frac{1}{2} c_d \rho_{\text{air}} A_{\text{cs}} |\mathbf{v}_{\text{rel}}| \mathbf{v}_{\text{rel}},$$

   where c_d is the drag coefficient (which depends on the shape of the plane). Write an expression for the power expended by the airplane to combat the drag force from the air.

   (b) Airplanes generate lift by deflecting air downward.

   i. Estimate the air mass per unit time which is deflected by the wings of the plain.

   ii. Estimate the power expended by the plane for lift.

   (c) Estimate the speed at which an airplane flies relative to the air by minimizing the power expended by the plane. To get a numeric answer, you may use the following parameters:

   $$m_{\text{plane}} \sim 8 \times 10^4 \text{ kg}, \quad \rho_{\text{air}} \sim 1 \text{ kg/m}^3, \quad c_d \sim 10^{-2}, \quad A_{\text{cs}} \sim 100 \text{ m}^2.$$

Next, we estimate the jet stream speed using flight times. Because the jet stream speed on Earth varies greatly with location, time of year, and climate effects (such as El Niño and La Niña), you will instead consider the fictitious planet Orb, where the jet stream is **eastward** and uniform in the region of interest. At the end of the problem is a map of the region, whose area is much smaller than the surface area of Orb (i.e., you can neglect the curvature of Orb).

2. At cruising altitude, we asume all airplanes travel at a fixed speed v_{rel} relative to the air. (This is not necessarily the same as your answer to 1(c), which was just a rough estimate.) Additionally, we assume that flights occur in three stages – (1) taxi and takeoff, (2) flight at cruising altitude, (3) landing and taxi – and that stages (1) and (3) take a fixed total time t_0 for every flight.

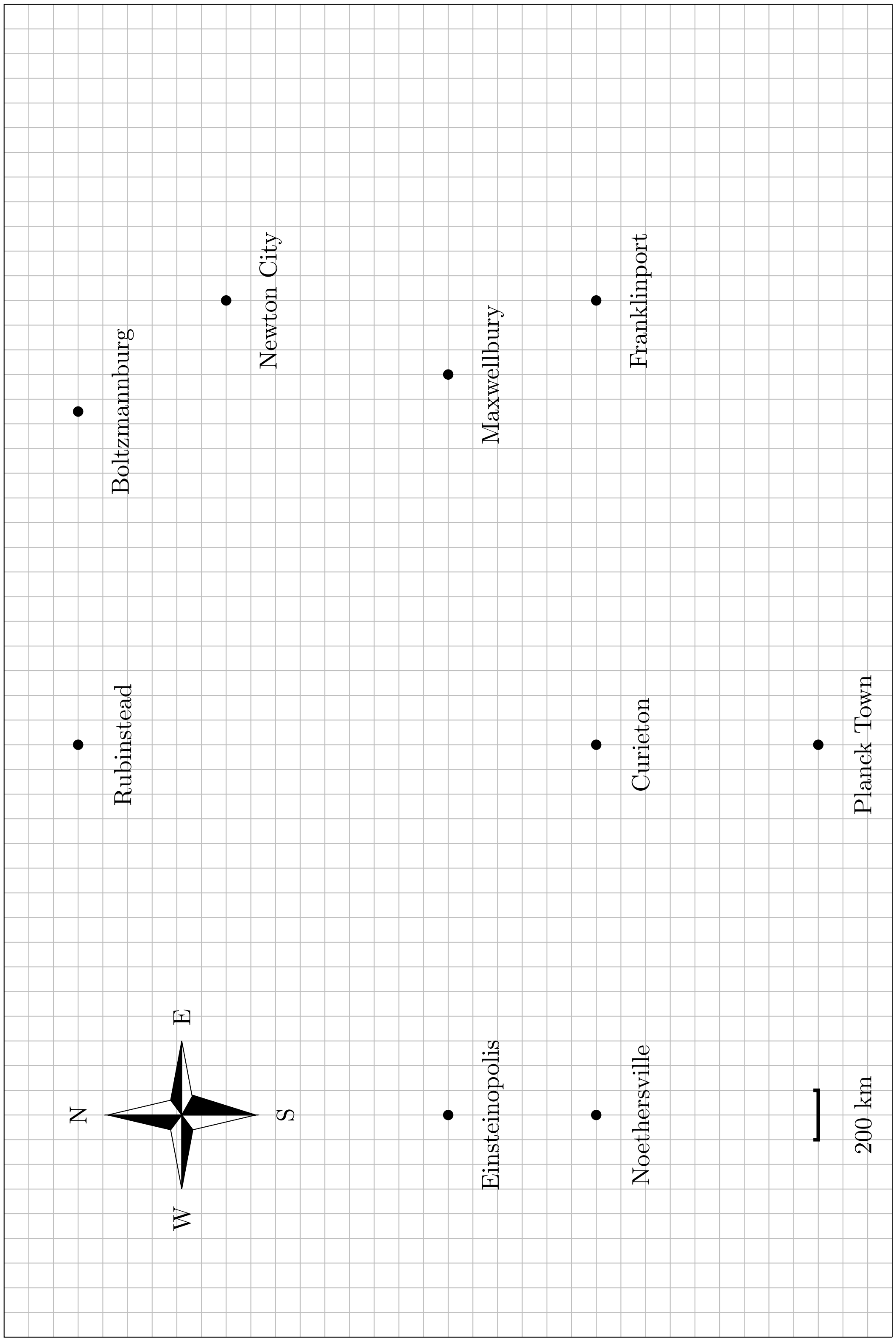
   (a) Suppose a plane, at cruising altitude, is traveling at an angle θ away from due east relative to the ground. What is the speed of the plane relative to the ground? Give your answer in terms of v_{rel}, θ, and v_w, the speed of the jet stream relative to the Earth's surface.

   (b) If the plane travels a distance D, what is the total travel time t, including taxi, takeoff, and landing?

(c) Below, we present some data on airplane flights on Planet Orb. Each of the flight times shown below has an independent uncertainty of $\Delta t = 5\,\text{min}$. From the data and the map, determine v_w and v_{rel}, giving your answers in km per hour with uncertainties. Indicate clearly what two quantities you are plotting against each other on each graph that you plot.

Departure City	**Arrival City**	t **(min)**
Noethersville	Rubinstead	185
Rubinstead	Noethersville	286
Curieton	Franklinport	107
Franklinport	Curieton	244
Planck Town	Maxwellbury	143
Maxwellbury	Planck Town	256
Rubinstead	Boltzmannburg	92
Boltzmannburg	Rubinstead	190
Einsteinopolis	Maxwellbury	160
Maxwellbury	Einsteinopolis	384
Planck Town	Franklinport	128
Franklinport	Planck Town	266
Einsteinopolis	Franklinport	188
Franklinport	Einsteinopolis	431
Boltzmannburg	Maxwellbury	135
Maxwellbury	Boltzmannburg	150

Departure City	**Arrival City**	t **(min)**
Noethersville	Einsteinopolis	68
Einsteinopolis	Noethersville	74
Franklinport	Newton City	144
Newton City	Franklinport	129
Curieton	Rubinstead	186
Rubinstead	Curieton	175
Planck Town	Curieton	95
Curieton	Planck Town	102
Planck Town	Rubinstead	249
Rubinstead	Planck Town	250



Boltzmannburg
Newton City
Maxwellbury
Franklinport
Rubinstead
Curieton
Planck Town
N
E
S
W
Einsteinopolis
Noethersville
200 km

Question 2

The Dark Forest

Dark matter could be made of hypothetical, extremely light particles called axions. Because individual axions are so light, experiments do not search for individual axions, but rather for the classical axion field formed by a large collection of axions, which oscillates as

$$a(t) = a_0 \sin(\omega t).$$

This is analogous to how a large collection of photons can form a classical electromagnetic field. In the presence of a magnetic field $\mathbf{B}$ and an axion field a, the axion field produces an effective current

$$\mathbf{J} = g\dot{a}\mathbf{B}$$

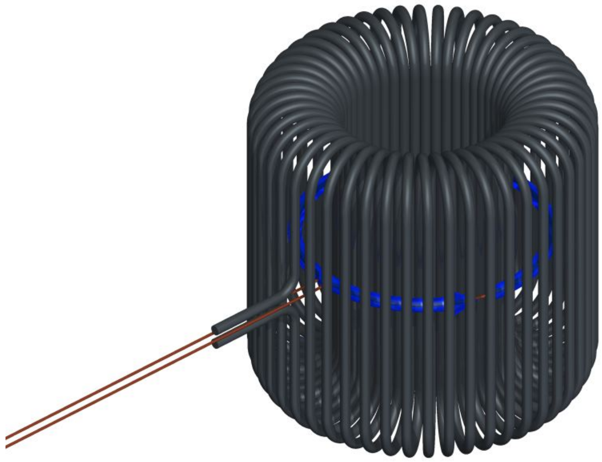
where we define $\dot{a} = da/dt$. The effective current produces electromagnetic fields in exactly the same way as ordinary current, though it does not come from the motion of actual charges. Experiments can search for axion dark matter using systems which are resonantly driven by this current.

You may use fundamental constants in your answers, such as

$$c = 3.00 \times 10^8\,\mathrm{m/s} \qquad \hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s} \qquad e = 1.602 \times 10^{-19}\,\mathrm{C}$$
$$G = 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2} \qquad \mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad k_B = 1.38 \times 10^{-23}\,\mathrm{J/K}.$$

You do not have to provide numeric answers unless asked. When asked to "estimate", you may drop constants of order one. The numeric values provided below are from standard references where $\hbar$, c, μ_0, and ϵ_0 are set to one; to get correct numeric results, you must restore these factors yourself.

1. First, we will describe some physical properties of the axion field.
   (a) Consider a single axion at rest, with mass m. Find its associated angular frequency ω. This will be the angular frequency of the corresponding classical field, when there are many axions.
   (b) Suppose dark matter is distributed spherically symmetrically in the galaxy with uniform density ρ. The solar system is a distance r from the center of the galaxy and orbits around it with period T. Neglecting everything besides dark matter, find the dark matter density ρ.
   (c) The energy density of the axion field is $m^2 a_0^2/(2\hbar^3 c)$. Find the axion field amplitude a_0.
   (d) The radius and period of the Sun's orbit, as well as a typical axion mass, are
   $$r = 2.5 \times 10^{20}\ \mathrm{m}, \quad T = 7.1 \times 10^{15}\ \mathrm{s}, \quad m = 1.0 \times 10^{-9}\ \mathrm{eV}.$$
   Numerically compute the axion field amplitude a_0.
   (e) In this problem, we treat the axion field as spatially uniform within a terrestrial laboratory. To verify that this assumption is reasonable, numerically estimate the axion field's wavelength λ, assuming the axions have the same galactic speed as the Sun.
   (f) In part (a), you found ω by neglecting the axion's speed. In reality, the axion's finite speed changes the frequency to $\omega + \Delta\omega$, in a frame at rest with respect to the galactic center. Numerically estimate $\Delta\omega/\omega$ to show that it is reasonable to neglect this effect.



The ABRACADABRA[1] experiment, currently taking data at MIT, is a toroidal solenoid with inner and outer radius R_{in} and R_{out} and height h. You may assume $h \gg R_{\text{out}}$ for simplicity. A superconducting wire carrying current I wraps N times around the toroid, where N is high enough to neglect the discreteness of the wires. A circular pickup loop with radius slightly less than R_{in} is placed at the center of the toroid.

2. Now, we will find the axion signal generated in the ABRACADABRA apparatus.

   (a) Find the magnetic field $\mathbf{B}(\mathbf{r})$ inside the toroid due to the superconducting current.

   (b) The superconducting wires lose their superconductivity when exposed to a magnetic field greater than $B_{\max}$. Find the maximum possible current $I_{\max}$ that can be used, and assume this current is used in later parts.

   (c) Assuming that ω is small, find the magnetic flux $\Phi_B(t)$ through the pickup loop due to the axion field in terms of a_0, g, ω, $B_{\max}$, and the dimensions of the apparatus. (You may ignore any currents induced on the surfaces of the superconducting wires. Accounting for them makes the problem much harder, but does not substantially affect the final result.)

   (d) If ω is too large, the result above breaks down due to radiation effects. Estimate the frequency ω_c where this happens.

   (e) Using the design values

   $$R_{\text{in}} = 0.5 \text{ m}, \quad R_{\text{out}} = 1.0 \text{ m}, \quad h = 2.0 \text{ m}$$

   estimate the numerical value of ω/ω_c.

   (f) Let Φ_0 be the amplitude of the time-varying axion flux. Using the typical values

   $$B_{\max} = 5.0 \text{ T}, \quad g = 1.0 \times 10^{-16} \text{ GeV}^{-1}$$

   and your previous results, compute the numerical value of Φ_0.

[1]aka, A Broadband/Resonant Approach to Cosmic Axion Detection with an Amplifying B-field Ring Apparatus.

The pickup loop has inductance L and is attached to a capacitor, forming a circuit with resonant frequency equal to the axion frequency ω. The circuit also has a small internal resistance R in series, and is at temperature T. The axion signal can be detected by monitoring the current in the circuit. The main source of noise is thermal noise, which causes fluctuations in the current.

3. We will now estimate the sensitivity of ABRACADABRA to axions.

   (a) The axion produces a current which oscillates sinusoidally. Find the signal current amplitude I_s in terms of ω, Φ_0, and the circuit parameters.

   (b) Find the average value of the current squared $\langle I^2 \rangle$ in the circuit due to thermal noise.

   (c) At any moment in time, the noise current is oscillating sinusoidally with typical amplitude $I_n = \sqrt{\langle I^2 \rangle}$, which is much larger than I_s. However, the phase of the noise current also fluctuates randomly, so that after a typical time t_c, its phase will be roughly independent of the phase it had before. Find an estimate for t_c in terms of ω and the circuit parameters. (Hint: at any given moment, the thermal noise current is simultaneously being produced by the random motion of electrons in the circuit, and damped by the resistor.)

   (d) Suppose the experiment runs for a total time $t_e \gg t_c$. Roughly estimate the average amplitude of the noise current over this period of time.

   (e) The axion is detectable if the signal current amplitude is larger than the averaged noise current amplitude, and the circuit parameters are

   $$L = 1 \text{ mH}, \quad R = 10 \text{ m}\Omega, \quad T = 0.1 \text{ K}.$$

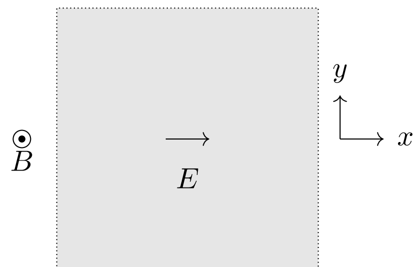
   Roughly numerically estimate the time needed to potentially detect the axion. (Hint: if your answer seems strange, note that in reality, the axion's phase also fluctuates over time, because of the effect of part 1(f). In addition, we don't know ω ahead of time, so the experiment needs to be run many times. We ignored these effects here to keep things simple.)

Question 3

Great Hall

The classical Hall effect was first measured by Edwin Hall in 1879, shortly after the publication of Maxwell's equations. In all parts of this problem, materials contain n_V electrons per unit volume, and each electron has charge $q_e < 0$ and mass m_e. You may use these quantities in all of your answers. We will begin by investigating the implications of the classical Hall effect.

1. An infinite plate in the xy plane, with thickness d in the z direction, is placed in a uniform magnetic field $\mathbf{B} = B\hat{z}$ as shown. An electric field $\mathbf{E} = E\hat{x}$ is applied in the plane of the plate and the system is allowed to reach a steady state.



(a) If the electrons have velocity $\mathbf{v}$ at steady state, what is the current density $\mathbf{J}$? Recall that $\mathbf{J}$ is defined as the total flow of charge through a unit cross-section area per unit time.

(b) In the Drude model, electrons are subject to both the Lorentz force and a damping force $-\gamma\mathbf{v}$, where γ is a constant that depends on the material. In the above system, what is the current density in the steady state? Give both the magnitude and direction of $\mathbf{J}$, e.g. in polar coordinates.

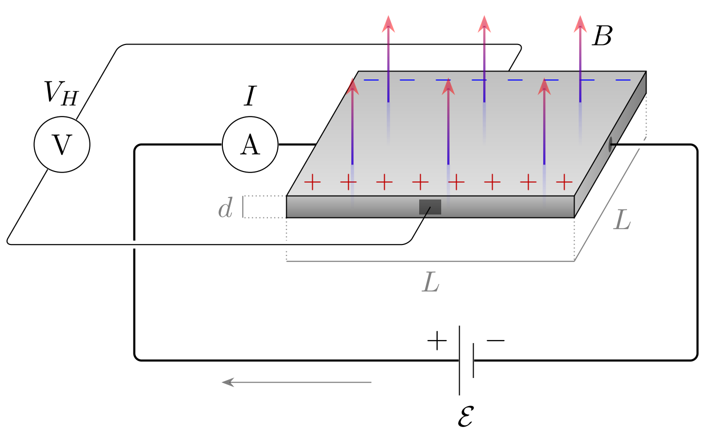
(c) Compute the electrical resistivity,

$$\rho_0 = \lim_{B\to 0} \frac{E}{|J_x|}$$

and the transverse Hall resistivity

$$\rho_H = \lim_{\gamma\to 0} \frac{E}{|J_y|}.$$

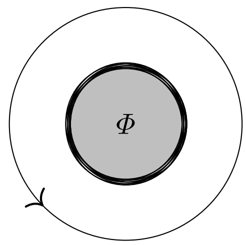
(d) A Hall effect sensor detects the strength of magnetic fields. Consider the following circuit consisting of a square plate of side length L and thickness d in a perpendicular uniform magnetic field B.



A longitudinal emf $\mathcal{E}$ is applied to the plate. At steady state, a Hall voltage V_H is measured across the plate due to the buildup of charge on either side of the plate. If the electrical resistivity of the plate at zero magnetic field is ρ_0, what is the Hall voltage V_H and the current I through the plate? Express your answer in terms of ρ_0, $\mathcal{E}$, B, and the dimensions of the plate.

Experiments in the 20th century revealed that in many materials, the Hall resistivity could only take certain discrete values. We will now show how this follows from Bohr quantization. (These next parts are independent of the first part of the problem.)

2. A zero-resistance loop of wire of radius R and cross-sectional area A_w carries a counterclockwise current I. A solenoid through the middle of the loops carries magnetic flux Φ out of the page, which we define to be the positive $\hat{z}$ direction.



   (a) If the electrons all have the same speed, what is the angular momentum of each electron?

   (b) If we allow the flux in the solenoid to change, the usual, "mechanical" angular momentum L of each electron is not conserved. Instead, a quantity called the canonical angular momentum, $L_{\text{can}} = L + Cq_e\Phi$, for some constant C, is conserved. Find C.

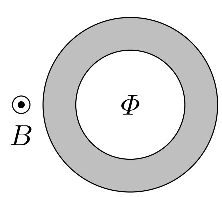
   (c) The Bohr quantization condition says that for a closed circular orbit, an integer number of de Broglie wavelengths must fit in its circumference. The de Broglie wavelength is

$$\lambda = \frac{h}{p_{\text{can}}},$$

   where h is Planck's constant, and $p_{\text{can}} = L_{\text{can}}/R$ is the canonical momentum. For a given solenoid flux Φ, what is the set of allowed mechanical angular momenta L?

   (d) What is the minimum possible change in the magnetic flux for which the same set of mechanical angular momenta is allowed? This is known as the flux quantum.

3. Now, consider an annulus held perpendicular to a fixed, uniform external magnetic field B, and suppose an additional, tunable magnetic flux Φ threads the center of the annulus, with both pointing out of the page. The annulus has a transverse Hall resistance R_H (i.e., an EMF of $\mathcal{E}$ around the annulus generates a perpendicular current $\mathcal{E}/R_H$ via the Hall effect) and you may neglect its self-inductance.



(a) Suppose Φ begins to increase slowly and steadily in time. After a short time, the electrons will begin flowing steadily from one side of the annulus to the other. Do the electrons move inward or outward? Justify your answer.

(b) If the threaded flux increases by $\Delta\Phi$, how many electrons pass from one edge of the annulus to the other? You may use R_H, among other variables, in your answer.

(c) As we showed in 2(d), if the magnetic flux changes by the flux quantum Φ_q, the allowed orbits from Bohr quantization are unchanged. Quantum mechanics thus tells us that in conventional materials, if the magnetic flux changes by Φ_q, an integer number k of electrons must pass from one edge to another. What constraint does this place on the Hall resistance?