General Relativity in the Undergraduate Physics Curriculum

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Einstein's general relativity is increasingly important in contemporary physics on the frontiers of both the very largest distance scales (astrophysics and cosmology) and the very smallest(elementary particle physics). This paper makes the case for a 'physics first' approach to introducing general relativity to undergraduate physics majors.

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I. INTRODUCTION

Einstein's 1915 relativistic theory of gravity — general relativity — will soon be a century old. It is the classical theory of one of the four fundamental forces. It underlies our contemporary understanding of the big bang, black holes, pulsars, quasars, X-ray sources, the final destiny of stars, gravitational waves, and the evolution of the universe itself. It is the intellectual origin of many of the ideas at play in the quest for a unified theory of the fundamental forces that includes gravity. The heart of general relativity is one of the most beautiful and revolutionary ideas in modern science — the idea that gravity *is* the geometry of curved four-dimensional spacetime. General relativity and quantum mechanics are usually regarded as the two greatest developments of twentieth-century physics.

Yet, paradoxically, general relativity — so well established, so important for several branches of physics, and so simple in its basic conception — is often not represented anywhere in the undergraduate physics curriculum. An informal survey by William Hiscock [1] of the course offerings of 32 mid-western research universities found only a handful that offered an intermediate (junior/senior) course in general relativity as part of the undergraduate physics curriculum. This has the consequence that many students see gravity first in the context of planetary orbits in basic mechanics and next, if at all, in an advanced graduate course designed in part for prospective specialists in the subject. There might have been an argument for such an organization half a century ago. But there is none today in an era when gravitational physics is increasingly important, increasingly topical, increasingly integrated with other areas of physics, and increasingly connected with experiment and observation. In the author's opinion, every undergraduate physics major should have an opportunity to be introduced to general relativity.

Its importance in contemporary physics is not the only reason for introducing undergraduates to general relativity. There are others: First, the subject excites interest in students. Warped spacetime, black holes, and the big bang are the focus both of contemporary research and of popular scientific fascination. Students specializing in physics naturally want to know more. A further argument for undergraduate general relativity is accessibility. As I hope to show in this paper, a number of important phenomena of gravitational physics can be efficiently introduced with just a basic background in mechanics and a minimum of mathematics beyond the usual advanced calculus tool kit. Other subjects of great contemporary importance such as high temperature superconductivity or gauge theories of the strong interactions require much more prerequisite information. General relativity can be made accessible to both students and faculty alike at the undergraduate level.

It is probably fruitless to speculate on why a subject as basic, accessible, and important as general relativity is not taught more widely as part of the undergraduate physics curriculum. Limited time, limited resources, inertia, tradition, and misconceptions may all play a role. Certainly it is not a lack of textbooks. Refs. [2]—[22] are a partial list of texts known to the author[1] that treat general relativity in some way at an introductory level.

Available time is one of the obstacles to introducing general relativity at the undergraduate level. The deductive approach to teaching this subject (as for most others) is to assemble the necessary mathematical tools, motivate the field equations, solve the equations in interesting circumstances, and compare the predictions with observation and experiment. This 'math first' order takes time to develop for general relativity which may not be available to either students or faculty. This article describes a different, 'physics first' approach to introducing general relativity at the junior/senior level. Briefly, the simplest physically relevant solutions to the Einstein equation are introduced *first*, without derivation, as curved spacetimes whose properties and observable consequences can be explored by a study of the motion of test particles and light rays. This brings the student to interesting physical phenomena as quickly as possible. It is the part of the subject most directly connected to classical mechanics and the part that requires a minimum of 'new' mathematical ideas. Later the Ein-

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[1] This list consists of texts known to the author, published after 1975, and judged to be introductory. It does not pretend to be either complete or selective, nor is it a representation that the texts are readily available.

stein equation can be motivated and solved to show where the solutions come from. When time is limited this is a surer and more direct route to getting at the applications of general relativity that are important in contemporary science.

Section II expands very briefly on the importance of general relativity in contemporary physics. Section III outlines the basic structure of the subject. Sections IV and V describe the 'math first' and 'physics first' approaches to introducing general relativity to undergraduate physics majors. This is not an even-handed comparison. The 'math first' approach is described only to contrast it with the 'physics first' approach which is advocated in this paper. Section VI illustrates how ideas from classical mechanics can be used to calculate important effects in general relativity. Section VII reports the personal experiences of the author in using the 'physics first' method.

II. WHERE IS GENERAL RELATIVITY IMPORTANT?

Gravity is the weakest of the four fundamental forces at accessible energy scales. The ratio of the gravitational force to the electric force between two protons separated by a distance r is (in Gaussian electromagnetic units)

$$\frac{F_{\rm grav}}{F_{\rm elec}} = \frac{Gm_p^2/r^2}{e^2/r^2} = \frac{Gm_p^2}{e^2} \sim 10^{-40}\,. \tag{2.1}$$

Gravity might thus seem to be negligible. But three other facts explain why it is important and where it is important. First, gravity is a universal force coupling to all forms of mass and energy. Second, gravity is a long-range force in contrast to the weak and strong forces which are characterized by nucleus-size ranges and below. Third, and most importantly, gravity is unscreened. There is no negative "gravitational charge"; mass is always positive.

These three facts explain why gravity is the dominant force governing the structure of the universe on the largest scales of space and time — the scales of astrophysics and cosmology. The strong and weak forces are short range. The relatively much greater strength of electromagnetic forces ensures that charges will be screened in an electrically neutral universe like ours. Only gravity is left to operate on very large scales.

Relativistic gravity — general relativity — is important for an object of mass M and size R when

$$q \equiv \frac{GM}{Rc^2} \sim 1\,. \tag{2.2}$$

Neutron stars $(q \sim .1)$ and black holes $(q \sim .5)$ are relativistic objects by this rough criterion. So is our universe $(q \sim 1)$ if we take R to be the present Hubble distance and M to be the mass within it. Figure 1 displays some phenomena for which relativity is important and ones for which it is not.

General relativity can sometimes be important even when q is small provided compensating observational precision can be achieved. For the Sun $q \sim 10^{-6}$, yet the solar system is the domain of the precision tests that confirm general relativity to as much as 1 part in 10^5 [23]. For the Earth $q \sim 10^{-9}$, yet

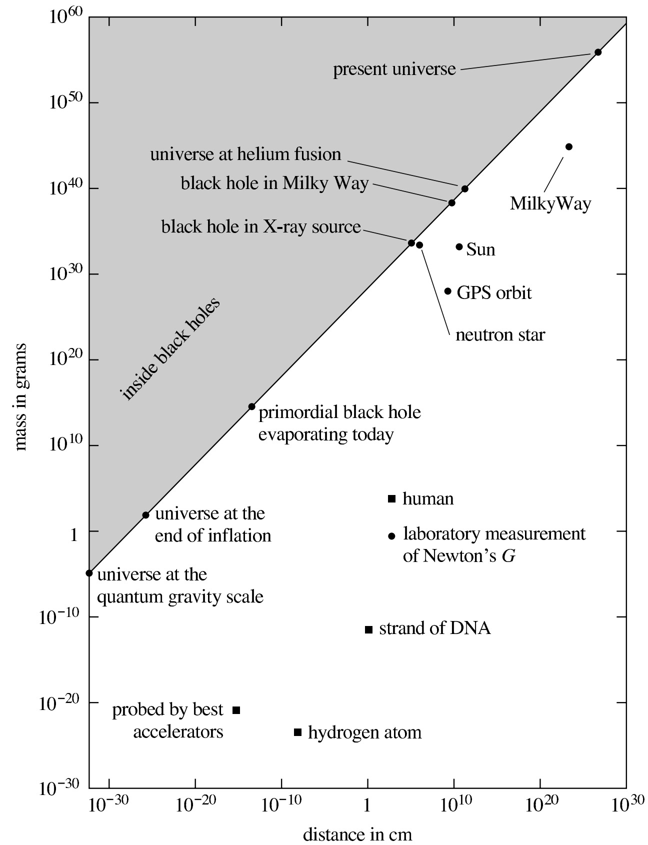

FIG. 1: Where relativistic gravity is important. This figure shows selected phenomena plotted on a graph of their characteristic mass M *vs.* their characteristic size R. The distance scale ranges from the smallest considered in today's physics (the Planck scale) to the largest (the size of the visible universe). Masses range from those of elementary particles to the mass in the visible universe. Phenomena above the line $2GM = c^2R$ are inside black holes and inaccessible. Phenomena on that line, or close to it and below, are the ones for which general relativity is important [*cf.* (2.2)] and are indicated by dots. Phenomena further away, for which relativity is unimportant, are indicated by squares. For the universe R is the Hubble distance and M the mass inside it. Our universe roughly evolves along the line $2GM = c^2R$ from the smallest scales of quantum gravity to the largest characterizing present cosmology. (Reproduced from [11] which was adapted from a figure prepared by C. Will for [25]).

general relativistic effects are important for the operation of the Global Positioning System (GPS) [24].

Relativistic gravity is also important on the smallest scales considered in contemporary physics — those of quantum gravity. These are characterized by the Planck length ℓ

$$\ell_{\rm Pl} = (G\hbar/c^3)^{\frac{1}{2}} \sim 10^{-33}\,{\rm cm}\,. \tag{2.3}$$

This is much, much smaller than even the scale of the strong interactions $\sim 10^{-13}$ cm. Yet, this is the scale which many contemporary explorers believe will naturally characterize the final theory unifying the four fundamental forces including gravity. This is the characteristic scale of string theory. This is the scale that will characterize the union of the two great developments of twentieth century physics — general relativity and quantum mechanics.

The important point for this discussion is that the last few decades have seen dramatic growth in observational data on the frontier of the very large, and an equally dramatic growth in theoretical confidence in exploring the frontier of the very small. Black holes, for example, are no longer a theorist's dream. They have been identified at the center of galaxies (including our own) and in X-ray binaries. They are central to the explanations of the most energetic phenomena in the universe such as active galactic nuclei. On even larger scales, it is now a commonplace observation that cosmology has become a data driven science. Cosmological parameters once uncertain by orders of magnitude have been determined to accuracies of 10% [26, 27].

Adventures into Planck scale physics may be mostly in the minds of theorists, but the quest for a unified theory of the fundamental forces including gravity is being pursued with impressive vigor and confidence by a large community. Indeed, at the big bang where large and small are one, we should eventually see direct evidence of Planck scale physics.

For these reasons general relativity is increasingly central to today's physics. It is increasingly topical, increasingly connected with experiment and observation, and increasingly integrated with other branches of physics[2]

III. KEY IDEAS IN GENERAL RELATIVITY

This section sketches a few key ideas in general relativity for those who may not be familiar with the theory. The intent is not to offer an exposition of these ideas. That, after all, is properly the task of texts on the subject. Rather, the purpose is merely to mention ideas that will occur in the subsequent discussion and to illustrate the simplicity of the conceptual structure of the subject.

It's potentially misleading to summarize any subject in physics in terms of slogans. However, the following three roughly stated ideas are central to general relativity.

- *Gravity is Geometry.* Phenomena familiarly seen as arising from gravitational forces in a Newtonian context are more generally due to the curvature of geometry of four-dimensional spacetime.

- *Mass-Energy is the Source of Spacetime Curvature.* Mass is the source of spacetime curvature and, since general relativity incorporates special relativity, any form of energy is also a source of spacetime curvature.

- *Free Mass Moves on Straight Paths in Curved Spacetime.* In general relativity, the Earth moves around the Sun in the orbit it does, not because of a gravitational force exerted by the Sun, but because it is following a straight path in the curved spacetime produced by the Sun.

Making these ideas more precise and more explicit is an objective of any course in general relativity. We mention a few steps toward this objective here.

Points in four-dimensional spacetime can be located by four coordinates $x^\alpha, \alpha = 0,1,2,3$. Coordinates are arbitrary provided they label points uniquely. Generally, several different coordinate patches are required to label all the points in spacetime.

The geometry of a spacetime is specified by giving the *metric*, $g_{\alpha\beta}(x)$, where the (x) indicates that the metric is generally a function of all four coordinates. The metric determines the squared distance ds^2 in four-dimensional spacetime between points separated by infinitesimal coordinate intervals dx^α. Specifically,

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta \tag{3.1}$$

where a double sum over α and β from 0 to 3 is implied. Integration of the ds specfied by this expression gives the distance along curves.

Metrics satisfy the *Einstein equation*

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = \frac{8\pi G}{c^4}\, T_{\alpha\beta} \tag{3.2}$$

relating a measure of curvature on the left hand side to the energy-momentum tensor of matter on the right. This Einstein equation comprises 10 non-linear, partial differential equations for the metric $g_{\alpha\beta}(x)$. An important example of a solution to the Einstein equation is the Schwarzschild geometry giving the metric in the empty space outside a spherically symmetric black hole or star. In standard Schwarzschild spherical coordinates $x^\alpha = (t,r,\theta,\phi)$ this is

$$\begin{aligned} ds^2 = &-\left(1-\frac{2GM}{c^2r}\right)(cdt)^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2 \\ &+ r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{3.3}$$

where M is the mass of the black hole or star. This, to an excellent approximation, describes the curved spacetime outside our Sun.

Test particles with masses too small to affect the ambient geometry move on straight paths in it. More precisely, they move between any two points, A and B, in spacetime on a world line (curve) of stationary proper time τ. The proper time along a world line is the distance along it measured in time units. Thus, $d\tau^2 = -ds^2/c^2$. (The negative sign is so $d\tau^2$ is positive along the world lines of particles which always move with less than the speed of light). Curves of stationary proper time are called *geodesics*.

The world line of a particle through spacetime from A to B can be described by giving its coordinates $x^\alpha(\lambda)$ as a function of any parameter that takes fixed values on the end points. For instance using the Schwarzschild metric (3.3), the principle of

[2] For more on the importance of general relativity in contemporary physics, see *e.g.* [25].

stationary proper time takes the form:

$$\delta \int_A^B d\tau = \delta \int_A^B d\lambda \frac{1}{c^2}\left[\left(1-\frac{2GM}{c^2 r}\right)(c\dot{t})^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)\right]^{\frac{1}{2}} = 0 \quad (3.4)$$

where a dot denotes a derivative with respect to λ and δ means the first variation as in classical mechanics.

The variational principle (3.4) for stationary proper time has the same form as the variational principle for stationary action in classical mechanics. The Lagrangian $L(\dot{x}^\alpha, x^\alpha)$ is the integrand of (3.4). Lagrange's equations are the geodesic equations of motion. Their form can be made especially simple by choosing proper time τ for the parameter λ. From them one can deduce the conservation of energy, the conservation of angular momentum, and an effective equation for radial motion in the Schwarzschild geometry as we will describe in Section VI [cf.(6.1)]. With that, one can calculate a spectrum of phenomena ranging from the precession of planetary orbits to the collapse to a black hole.

When extended to light rays and implemented in appropriate metrics, the geodesic equations are enough to explore most of the important applications of relativistic gravity displayed in Table 1.

IV. TEACHING GENERAL RELATIVITY — MATH FIRST

The deductive approach to teaching many subjects in physics is to

1. Introduce the necessary mathematical tools;
2. Motivate and explain the basic field equations;
3. Solve the field equations in interesting circumstances;
4. Apply the solutions to make predictions and compare with observation and experiment.

For electromagnetism, (1)–(4) are, *e.g.* (1) Vector calculus, (2) Maxwell's equations, (3) boundary value problems, the fields of point particles, radiation fields, etc., (4) charged particle motion, circuits, wave guides and cavities, antennas, dielectric and magnetic materials, magnetohydrodynamics — a list that could very easily be extended. For gravitation (1)–(4) are *e.g.* (1) differential geometry, (2) the Einstein equation and the geodesic equation, (3) the solutions for spherical symmetry, cosmological models, gravitational waves, relativistic stars, etc. Table 1 lists some of the applications of general relativity that constitute (4).

This deductive order of presentation is logical; it is the order used by the great classic texts [28, 29, 30, 31]; and it is the order used in standard graduate courses introducing the subject at an advanced level. But the deductive order does have some drawbacks for an elementary introduction to physics majors in a limited time.

Differential geometry is a deep and beautiful mathematical subject. However, even an elementary introduction to its basic ideas and methods are not a part of the typical advanced calculus tool kit acquired by physics majors in their first few years. This is 'new math'. In contrast, the vector calculus central to electromagnetism *is* part of this tool kit.

It is possible at the undergraduate level to give an introduction to the basic mathematical ideas of manifolds, vectors, dual vectors, tensors, metric, covariant derivative, and curvature. Indeed, many students feel empowered by learning new mathematics. But it does take time. It also must be practiced. The author's experience is that many students at this level need considerable exercise before they are able to accurately and efficiently manipulate tensorial expressions and feel at home with the four-dimensional mathematical concepts necessary to formulate Einstein's equation. When time is limited, pursuing the deductive order may leave little available for the interesting applications of general relativity.

Further, solving the Einstein equation to exhibit physically relevant spacetime geometries is a difficult matter. Their nonlinear nature means that there is no known general solution outside of linearized gravity. Each new situation, *e.g.* spherical symmetry, homogeneous and isotropic cosmological models, gravitational plane waves, is typically a new problem in applied mathematics. Deriving the solutions only adds to the time expended before interesting applications can be discussed.

Many of the successful introductory texts in general relativity follow this logical order at various levels of compromise. In the author's opinion, an outstanding example is Bernard Schutz's classic *A First Course in General Relativity* [19]. In the next section we consider a different way of introducing general relativity to undergraduates.

V. TEACHING GENERAL RELATIVITY — PHYSICS FIRST

Electricity and magnetism are not usually presented in introductory (freshman) courses in the deductive order described in the previous section. Specifically, we do not usually first develop vector calculus, then exhibit Maxwell's equations, then solve for the fields of charges, currents, and radiation, and finally apply these to realistic electromagnetic phenomena. Rather, the typical course posits the fields of the simplest physically relevant examples, for instance the electric field of a point charge, the magnetic field of a straight wire, and the electromagnetic plane wave. These are used to build understanding of fields and their interaction with charges for immediate application to demonstrable electromagnetic phenomena. Maxwell's ten partial differential equations and their associated gauge and Lorentz invariances are better appreciated later, usually in a more advanced course.

General relativity can be efficiently introduced at an intermediate (junior/senior) level following the same 'physics first' model used in introducing electromagnetism. Specifically:

1. Exhibit the simplest physically important spacetime geometries *first*, without derivation;

Table 1
Some Important Applications of General Relativity

Global Positioning System (GPS)	Lense-Thirring precession of a gyroscope
Gravitational redshift	Cosmological redshift
Bending of light by the Sun	Expansion of the universe
Precession of Mercury's perihelion	Big-bang
Shapiro time delay	Cosmic Background Radiation
Gravitational lensing	The fate of the universe
Accretion disks around compact objects	Propagation of gravitational waves
Determining parameters of binary pulsars	Operation of gravitational wave detectors
Spherical gravitational collapse	X-ray sources
Formation of black holes	Active Galactic Nuclei
Hawking radiation from black holes	Neutron stars
Frame-dragging by a rotating body	

2. Derive the predictions of these geometries for observation by a study of the orbits of test particles and light rays moving in them;
3. Apply these predictions to realistic astrophysical situations and compare with experiment and observation;
4. Motivate the Einstein equation and solve it to show where the spacetime geometries posited in (1) come from.

These are essentially the same four elements that comprise the deductive approach described in the previous section, but in a different order. That order has considerable advantages for introducing general relativity at an intermediate level as we now describe:

A. Indications

Less 'New Math' Up Front: To exhibit a spacetime geometry, the only 'new math' ideas required are the metric and its relation to distances in space and time. To analyze the motion of test particles in these geometries, only the notions of four-vectors and geodesics are needed. These three new mathematical ideas are enough to explain in detail a wide range of physical phenomena, such as most of those in Table 1. Further, these three new mathematical ideas are among the simplest parts of a relativist's tool kit to introduce at an intermediate level. Four-vectors are often familiar from special relativity. Geodesics viewed as curves of extremal proper time are special cases of Lagrangian mechanics. The idea of a metric can be motivated from the theory of surfaces in three-dimensional flat space. A general theory of tensors as linear maps from vectors into the real numbers is not required because only one tensor — the metric — is ever used.

The Simplest Spacetimes are the Most Physically Relevant:

- The Sun is approximately spherical.
- Spherical black holes exhibit many characteristic properties of the most general black hole.
- The universe is approximately homogeneous and isotropic on scales above 100 megaparsecs.
- Detectable gravitational waves are weak.

These four facts mean that the simplest solutions of the Einstein equation are the ones most relevant for experiment and observation. The static, spherically symmetric Schwarzschild geometry (3.3) describes the solar system experimental tests, spherically symmetric gravitational collapse, and spherical black holes. The exactly homogeneous and isotropic Friedman-Robertson-Walker (FRW) models provide an excellent approximation to the structure and evolution of our universe from the big bang to the distant future. The linearized solutions of the Einstein equation about a flat space background describe detectable gravitational waves. A 'physics first' treatment of general relativity that concentrates on the simplest solutions of the Einstein equation is thus immediately relevant for physically realistic and important situations.

No Stopping before Some Physics: Students with different levels of experience, preparation, abilities, and preconceptions will take different lengths of time to acquire the basic concepts of general relativity. Beginning with the applications guarantees that, wherever the course ends, students will have gained some understanding of the basic physical phenomena (Table 1) which make general relativity so important and not simply of a mathematical structure which is the prerequisite for a deductive approach.

More Concrete, Less Abstract: Beginning with the applications rather than the abstract structure of the theory is easier for some students because it is more concrete. Beginning with the applications is also a surer way of driving home that general relativity is a part of physics whose predictions can be observationally tested and not a branch of mathematics.

Fewer Compromises: The analysis of the motion of test particles and light rays in the simplest geometries can be carried out in essentially the same way as it is done in advanced textbooks. No compromises of method or generality are needed.

Flexibility in Emphasis: Beginning with applications allows enough time to construct courses with different emphases; for instance on black holes, gravitational waves, cosmology, or experimental tests.

Closer to the Rest of the Undergraduate Physics Curriculum: Calculations of the orbits of test particles and light rays to explore curved spacetimes are exercises in mechanics. The symmetries of the simplest important solutions imply conservation laws. These can be used to reduce the calculation of orbits to one-dimensional motion in effective potentials. Even the content of the Einstein equation can be put in this form for simple situations. This allows the intuition and techniques developed in intermediate mechanics to be brought to bear, both for qualitative understanding and quantitative prediction. Conversely, this kind of example serves to extend and reinforce an understanding of mechanics. Indeed, in the author's experience a few students are surprised to find that mechanics is actually useful for something. The examples discussed in the next section will help illustrate the close connection between calculating geodesics and undergraduate mechanics.

Fewer Prerequisites: The close connection with mechanics described above and in the next section means that the only essential prerequisite to a 'physics first' exposition of general relativity is an intermediate course in mechanics. Neither quantum mechanics nor electrodynamics are necessary.Some acquaintance with special relativity is useful, but its brief treatment in many first year courses means that it is usually necessary to develop it *de novo* at the beginning of an introductory course in general relativity. Intermediate mechanics is thus the single essential physics prerequisite. This means that an introductory 'physics first' course in general relativity can be accessible to a wider range of physics majors at an earlier stage than courses designed to introduce students to other frontier areas.

Closer to Research Frontiers: Beginning with the applications means that students are closer sooner to the contemporary frontiers of astrophysics and particle physics that they can hear about in the seminars, read about in the newspapers, and see on popular television programs.

More Opportunities for Undergraduate Participation in Research: The applications of general relativity provide a broad range of topics for students to pursue independent study or even to make research contributions of contemporary interest. More importantly, it is possible to identify problems from the applications that are conceptually and technically accessible to undergraduate physics majors and can be completed in the limited time frame typically available. Problems that involve solving for the behavior of test particles, light rays, and gyroscopes are examples, as are questions involving linear gravitational waves, black holes, and simple cosmological models. The 'physics first' approach to teaching general relativity enables undergraduate participation in research because it treats such applications first.

Specialized Faculty Not Needed: Learning a subject while teaching it, or learning it better, is a part of every physics instructor's experience. The absence of previous instruction means that teaching an introductory general relativity course will often be the first exposure to the subject for many faculty. The process of learning by faculty is made easier by a 'physics first' approach for the same reasons it is easier for students. A wider variety of faculty will find this approach both familiar and manageable. Specialists in gravitational physics are not necessary.

B. Counterindications

No approach to teaching is without its price and the 'physics first' approach to introducing general relativity is no exception. The obvious disadvantage is that it does not follow the logically appealing deductive order, although it can get to the same point in the end. A 'physics first' approach to introducing general relativity *may* not be indicated in at least two circumstances: First, when there is enough time in the curriculum and enough student commitment to pursue the deductive order; second, when students already have significant preparation in differential geometry. Even then however it is a possible alternative. A 'physics first' approach is probably not indicated when the mathematics is of central interest, as for students concentrating in mathematics, and for physics students who study Einstein's theory mainly as an introduction to the mathematics of string theory.

VI. PARTICLE ORBITS OUTSIDE A SPHERICAL STAR OR BLACK HOLE

This section illustrates how the effective potential method developed in typical undergraduate mechanics courses can be applied to important problems in general relativity. Of the several possible illustrations, just one is considered here — the relativistic effects on test particle orbits outside a spherical star or black hole. Applications of this include the precession of perihelia of planets in the solar system and the location of the innermost stable circular orbit (ISCO) in an accretion disk powering an X-ray source. These examples also serve to illustrate how near the calculations of such effects are to starting principles in a 'physics first' approach to general relativity.

Nothing more than sketches of the calculations are intended. For more detail and the precise meaning of any quantities involved, the reader should consult any standard text[3].

The Schwarzschild geometry specified in (3.3) describes the curved spacetime outside a static, spherically symmetric body of mass M. This is the geometry outside the Sun to an excellent approximation, and is the geometry outside a spherical black hole. The motion of test particles is specified by the principle of stationary proper time in (3.4). The argument in

[3] We follow, with one minor simplifying exception, the notation in [11].

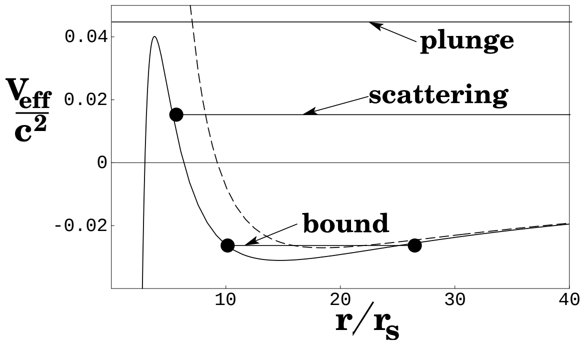

FIG. 2: The effective potential $V_{\text{eff}}(r)$ for the radial motion of particles in the Schwarzschild geometry outside a spherical star or black hole. The solid line shows $V_{\text{eff}}(r)/c^2$ defined in (6.1b) plotted against r/r_s where $r_s = GM/c^2$. The value of the angular momentum parameter is $\ell = 4.3GM/c^2$. The dotted line shows the Newtonian potential (the first two terms in (6.1b)) for the same value of ℓ. The qualitative behavior of the orbits depends on the value of the energy parameter $\mathcal{E}$. Values are shown for plunge orbits, scattering orbits, and bound orbits such as those executed by the planets in their motion about the Sun.

the square bracket of that equation can be thought of as a Lagrangian $L(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}, r)$. Lagrange's equations are the geodesic equations.

Time translation invariance implies a conserved energy per unit rest mass $\mathcal{E}$ related to $\partial L/\partial \dot{t}$. Spherical symmetry implies a conserved angular momentum per unit rest mass [4] ℓ for orbits in the equatorial plane which is proportional to $\partial L/\partial \dot{\phi}$. A third integral[5] of the motion $L = -1$. arises just from the definition of proper time. This integral can be combined with the other to to find an effective energy integral for the radial motion:

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \tag{6.1a}$$

where

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{c^2 r^3} \, . \tag{6.1b}$$

Here, r is the Schwarzschild radial coordinate of the test particle and τ is the proper time along its world line. The mass of the test particle is absent from these expressions. It cancels out because of the equality of gravitational and inertial mass. Eq. (6.1a) has the same form as the energy integral for a Newtonian central force problem. The first two terms in the effective potential (6.1b) have the same form as a Newtonian gravitational potential with a Newtonian centrifugal barrier. The third term provides a general relativistic correction to the Newtonian effective potential. Figure 2 shows its effects.

Circular orbits illustrate the importance of these effects. Newtonian gravity permits only one stable circular orbit for each ℓ. But in general relativity there are *two* circular orbits for values of ℓ such at that used in Figure 2. There is a stable circular orbit at the *minimum* of the effective potential such as one approximating the orbit of the Earth in its progress around the Sun. In addition there is an unstable circular orbit at the radius of the *maximum* of the effective potential.

The radii of the stable circular orbits are easily found from (6.1b):

$$r_{\text{stab.circ.}} = \frac{\ell^2}{2GM}\left\{1 + \left[1 - 12\left(\frac{GM}{c\ell}\right)^2\right]^{\frac{1}{2}}\right\}. \tag{6.2}$$

For sufficiently small ℓ there are *no* stable circular orbits. That is because the effective potential (6.1b) is everywhere attractive for low ℓ. In contrast to Newtonian physics, general relativity therefore implies that there is an *innermost* (smallest r) *stable circular orbit* (ISCO) whose radius is

$$r_{\text{ISCO}} = 6GM/c^2 \tag{6.3}$$

which is 1.5 times the characteristic radius of the black hole $r_s = 2GM/c^2$.

The ISCO is important for the astrophysics of black holes. The spectra of X-ray sources exhibit lines whose observed shapes can in principle be used to infer the properties of the black hole engine [32]. The shape of the line is determined by several factors but importantly affected by the gravitational redshift. That is maximum for radiation from parts of the accretion disk closest to the black hole, ie from the ISCO. This defines the red end of the observed line[6].

Another important prediction of general relativity derivable from the effective potential is the shape of bound orbits such as those of the planets. The shape of an orbit in the equatorial plane ($\theta = \pi/2$) may be specified by giving the azimuthal angle ϕ as a function of r. The orbits close if the total angle $\Delta\phi$ swept out in the passage away from the inner turning point and back again is 2π. This can be calculated from (6.1) and the angular momentum integral $\ell = r^2(d\phi/d\tau)$. Writing $d\phi/dr = (d\phi/d\tau)/(dr/d\tau)$ and using these two relations gives an expression for $d\phi/dr$ as a function of r, $\mathcal{E}$, and ℓ which can be integrated. The result is

$$\Delta\phi = 2\int dr(\ell/r^2)\left[2\left(\mathcal{E} - V_{\text{eff}}(r)\right)\right]^{-1/2} \tag{6.4}$$

where the integral is from the radius of the inner turning point to the outer one. When the relativistic term in (6.1b) is absent, $\Delta\phi = 2\pi$ for all $\mathcal{E}$ and ℓ. That is the closing of the Keplerian ellipses of Newtonian mechanics. The relativistic correction

[4] In [11], $\mathcal{E}$ and ℓ are defined as an energy and angular momentum per unit rest energy rather than per unit rest mass as here. With that definition of $\mathcal{E}$ both terms on the right hand side of (6.1a) would be divided by c^2. That is the minor exception alluded to in the previous footnote.

[5] 'Integral' here is used in the sense of classical mechanics, not in the sense of the inverse of differentiation.

[6] Realistic black holes are generally rotating, but the analysis then is not qualitatively different from that for the non-rotating Schwarzschild black hole.

to $V_{\rm eff}$ makes the orbit precess by small amount $\delta\phi = \Delta\phi - 2\pi$ on each pass. To lowest order in $1/c^2$ this is

$$\delta\phi = 6\pi \left(\frac{GM}{c\ell}\right)^2 . \tag{6.5}$$

In the solar system the precession is largest for Mercury but still only 43″ per century. The confirmation of that prediction [33] is an important test of general relativity.

The purpose of this section was not to explain or even emphasize the two effects of general relativity on particle orbits that were described here. Rather it was to show three things. First, that a standard technique developed in undergraduate mechanics can be used to calculate important predictions of general relativity. The second purpose was to show how close these important applications can come to starting principles in a 'physics first' approach to teaching general relativity. Introduce the Schwarzschild geometry (3.3), use the principle of stationary proper time (3.4) to find the geodesic equations or their integrals (6.1), use the effective potential method to qualitatively and quantitatively understand important properties of the orbits e.g. (6.3) and (6.4). That is just three steps from the basic ideas of metric and geodesics to important applications. Third, both of the applications treated here can be immediately related to contemporary observation and experiment.

Particle orbits in the Schwarzschild geometry are not the only important problems in which the effective potential method is useful. Motion in the geometry of a rotating black hole, the motion of test light rays in the Schwarzschild geometry, and the evolution of the Friedman-Robertson-Walker cosmological models are further examples where it can be usefully applied.

VII. CONCLUSION

A one quarter ($\sim$ 28 lectures) 'physics first' course in general relativity has been a standard junior/senior elective for physics majors at the University of California, Santa Barbara for approximately thirty years. In the limited span of a quarter the author is usually able to review special relativity, motivate gravity as geometry, derive the orbits in the Schwarzschild geometry in detail, describe the experimental tests, introduce black holes, and develop the Friedman-Robertson-Walker cosmological models. A semester provides more opportunities.

At Santa Barbara this course is routinely taught by faculty from many different areas of physics — general relativity of course, but also elementary particle physics and astrophysics. It has been taught by both theorists and experimentalists. For some of these colleagues teaching this course was their first serious experience with general relativity. They usually report that they were successful and enjoyed it.

In the author's experience students are excited by general relativity and motivated to pursue it. Often it is their first experience with a subject directly relevant to current research. It is one of the few contemporary subjects that can be taught without quantum mechanics or electromagnetism.

The author has written a text based on the 'physics first' approach [11] which comes with a solutions manual available to instructors for the approximately 400 problems of graded levels of difficulty.

'Physics first' is not the only way of introducing general relativity to undergraduate physics majors, but it works.

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[1] W. Hiscock, in a talk presented at the 2005 April Meeting of the APS, Tampa, FL, April 16, 2005.
[2] M. Berry, *Principles of Cosmology and Gravitation*, Institute of Physics, Bristol, UK and Philadelphia (1993).
[3] W. Burke, *Spacetime, Geometry, Cosmology*, University Science Books, Mill Valley (1980).
[4] J. Callahan, *The Geometry of Spacetime: An Introduction to Special and General Relativity*, Springer, New York (2000).
[5] T.-P. Cheng, *Relativity, Gravitation, and Cosmology: A Basic Introduction*, Oxford, New York (2005).
[6] C. Clarke, *Elementary General Relativity*, Edward Arnold, London, UK (1979) and Wiley, New York (1980).
[7] G.F.R. Ellis and R.M. Williams, *Flat and Curved Spacetime*, Clarendon Press, Oxford, UK (1988).
[8] J. Foster and J.D. Nightiingale, *A Short Course in General Relativity*, Springer-Verlag, New York (1995).
[9] R. Geroch, *General Relativity from A to B*, University of Chicago Press, Chicago (1978).
[10] R. Hakim, *Introduction to Relativistic Gravitation*, Cambridge University Press, Cambridge, UK (1999).
[11] J.B. Hartle, *Gravity*, Addison-Wesley, San Francisco (2003).
[12] L. Hughston and K.P. Tod, *An Introduction to General Relativity*, Cambridge University Press, Cambridge (1990).
[13] R. D'Inverno, *Introducing Einstein's Relativity*, Clarendon Press, New York and Oxford University Press, Oxford, UK (1995).
[14] I.R. Kenyon, *General Relativity*, Oxford University Press, Oxford, UK and New York (1990).
[15] M. Ludvigsen, *General Relativity: A Geometric Approach*, Cambridge University Press, Cambridge, UK and New York (1999).
[16] J.L. Martin, *General Relativity: A Guide to its Consequences for Gravity and Cosmology*, E. Horwood, Chichester, UK and Halsted Press, New York (1988).
[17] H. O'Hanian and R. Ruffini, *Gravitation and Spacetime*, Norton, New York (1994).

[18] W. Rindler, *Essential Relativity: special, general, and cosmological*, Springer-Verlag, New York (1977).

[19] B.F. Schutz, *A First Course in General Relativity*, Cambridge University Press, Cambridge, UK (1985).

[20] B.F. Schutz, *Gravity From the Ground Up*, Cambridge University Press, Cambridge, UK and New York (2003).

[21] E.F. Taylor and J.A. Wheeler, *Exploring Black Holes*, Addison-Wesley Longman, San Francisco (2000).

[22] P. Tourrenc, *Relativity and Gravitation*, Cambridge University Press, Cambridge, UK (1997).

[23] e.g. B. Bertotti B.L. Less, P. Tortora, *A Test of General Relativity Using Radio Links with the Cassini Spacecraft*, *Nature* **425**, 374-376 (2003).

[24] N. Ashby, *Relativity and the global positioning system*, *Phys. Today* **55**, no 5, 41–47 (2002).

[25] J. Hartle, *et al.*, *Gravitational Physics: Exploring the Structure of Space and Time*, (National Academy Press, Washington, DC, 1999).

[26] W. Freedman, B. Madore, *et al.*, *Final Results from The Hubble Space Telescope Project to Measure the Hubble Constant*, *Ap. J.* **553**, 47 (2001).

[27] D.N. Spergel *et al.*, *First year Wilkinson microwave anisotropy probe (WMAP) observations: Determination of cosmological parameters*, *Ap. J. Suppl.* **148**, 175–194 (2003); J.L. Tonry, *et al.*, *Cosmological results from high-z supernovae*, *Ap. J.* **594**, 1–24 (2003).

[28] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, London (1962).

[29] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, W.H. Freeman, San Francisco (1970).

[30] R. Wald, *General Relativity*, University of Chicago Press, Chicago (1984).

[31] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York (1972).

[32] See, e.g. C.S. Reynolds and M.A. Nowak, *Fluorescent Iron Lines as a Probe of Astrophysical Black Hole Systems*, *Physics Reports* **377**, 389 (2003); A.C. Fabian, K. Iwasawa, C.S. Reynolds, A.J. Young, *Broad Iron Lines in Active Galactic Nuclei*, *Publications of the Astronomical Society of the Pacific* **112**, 1145 (2000).

[33] I.I. Shapiro, *Solar System Tests of General Relativity*, in *General Relativity and Gravitation: 12th International Conference on General Relativity and Gravitation* ed by N. Ashby, D.F. Bartlett, and W. Wyss, (Cambridge University Press, Cambridge, 1990), 313-330.