AAPT UNITED STATES PHYSICS TEAM
AIP 2011

Semifinal Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts.
- Part A has four questions and is allowed 90 minutes.
- Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 3 parts: the cover sheet (page 2), Part A (pages 3-4), and Part B (pages 6-7). Examinees should be provided parts A and B individually, although they may keep the cover sheet.
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after March 31, 2011.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.
- **Please provide the examinees with graph paper for Part A.**

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Semifinal Exam

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your AAPT ID number, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  AAPT ID #
  Doe, Jamie
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 1, 2011.**

Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8\ \mathrm{N/kg}$	$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$	$k_\mathrm{m} = \mu_0/4\pi = 10^{-7}\ \mathrm{T \cdot m/A}$
$c = 3.00 \times 10^8\ \mathrm{m/s}$	$k_\mathrm{B} = 1.38 \times 10^{-23}\ \mathrm{J/K}$
$N_\mathrm{A} = 6.02 \times 10^{23}\ (\mathrm{mol})^{-1}$	$R = N_\mathrm{A} k_\mathrm{B} = 8.31\ \mathrm{J/(mol \cdot K)}$
$\sigma = 5.67 \times 10^{-8}\ \mathrm{J/(s \cdot m^2 \cdot K^4)}$	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
$1\mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$	$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}$
$m_e = 9.109 \times 10^{-31}\ \mathrm{kg} = 0.511\ \mathrm{MeV/c^2}$	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

Single bubble sonoluminescence occurs when sound waves cause a bubble suspended in a fluid to collapse so that the gas trapped inside increases in temperature enough to emit light. The bubble actually undergoes a series of expansions and collapses caused by the sound wave pressure variations.

We now consider a simplified model of a bubble undergoing sonoluminescence. Assume the bubble is originally at atmospheric pressure $P_0 = 101$ kPa. When the pressure in the fluid surrounding the bubble is decreased, the bubble expands isothermally to a radius of 36.0 μm. When the pressure increases again, the bubble collapses to a radius of 4.50 μm so quickly that no heat can escape. Between the collapse and subsequent expansion, the bubble undergoes isochoric (constant volume) cooling back to its original pressure and temperature. For a bubble containing a monatomic gas, suspended in water of $T = 293$ K, find

a. the number of moles of gas in the bubble,

b. the pressure after the expansion,

c. the pressure after collapse,

d. the temperature after the collapse, and

e. the total work done on the bubble during the whole process.

You may find the following useful: the specific heat capacity at constant volume is $C_V = 3R/2$ and the ratio of specific heat at constant pressure to constant volume is $\gamma = 5/3$ for a monatomic gas.

Solution

We consider the bubble to be filled with an ideal monatomic gas, so originally: $P_0V_0 = nRT_0$.

The bubble undergoes 3 processes: 1) isothermal expansion, 2) adiabatic collapse (no heat escapes), and 3) isochoric (constant volume) cooling. The final process is isochoric, so we know that the bubble's collapsed volume is equal to its original volume, so

$$V_2 = V_0,$$

and

$$P_0V_0 = P_0V_2 = nRT_0.$$

Rearranging,

$$n = \frac{P_0V_2}{RT_0}$$

$$n = \frac{P_0\frac{4}{3}\pi r_2^3}{RT_0}$$

$$n = \frac{101,000\frac{\text{N}}{\text{m}^2}\cdot\frac{4}{3}\pi\cdot(4.50\times10^{-6}\text{ m})^3}{8.31\frac{\text{J}}{\text{mol}\cdot\text{K}}\cdot 293\text{ K}} = \frac{3.86\times10^{-11}}{2430}\text{moles}$$

a) $n = 1.58 \times 10^{-14}$ moles.

Process 1: Isothermal expansion

This process is isothermal, so $T_1 = T_0$ and

$$P_1V_1 = nRT_1 = nRT_0$$

$$P_1 = \frac{nRT_0}{V_1} = \frac{1.58 \times 10^{-14} \text{ moles} \cdot 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}} \cdot 293 \text{ K}}{\frac{4}{3}\pi \cdot (3.60 \times 10^{-5} \text{ m})^3}$$

b) $P_1 = 197 \frac{\text{N}}{\text{m}^2} = 197$ Pa.

The work done *by* the bubble is:

$$W_1 = nRT_0 \ln \frac{V_1}{V_0}$$

$$W_1 = 1.58 \times 10^{-14} \text{ moles} \cdot 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}} \cdot 293 \text{ K} \cdot \ln\left(\frac{(3.60 \times 10^{-5})^3}{(4.50 \times 10^{-6})^3}\right)$$

$$W_1 = 2.40 \times 10^{-10} \text{ J}$$

So, the work done *on* the bubble during the expansion is:

$$W_1 = -2.40 \times 10^{-10} \text{ J}.$$

Process 2: Adiabatic collapse

For an adiabatic process

$$P_1V_1^\gamma = P_2V_2^\gamma$$

$$P_2 = \frac{P_1V_1^\gamma}{V_2^\gamma}$$

For a monatomic gas $\gamma = 5/3$ so,

$$P_2 = \frac{197 \frac{\text{N}}{\text{m}^2} \cdot (3.60 \times 10^{-5} \text{ m})^5}{(4.50 \times 10^{-6} \text{ m})^5}$$

c) $P_2 = 6.46 \times 10^6$ Pa.

And

$$T_2 = \frac{P_2V_2}{nR}$$

$$T_2 = \frac{6.46 \times 10^6 \text{ Pa} \cdot \frac{4}{3}\pi \cdot (4.50 \times 10^{-6} \text{ m})^3}{1.58 \times 10^{-14} \text{ moles} \cdot 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}}}$$

d) $T_2 = 18800$ K. *Lord, have mercy! That's hot!*

The work done *by* the bubble during an adiabatic process is

$$W_2 = -\Delta E_{internal} = -nC_v\Delta T$$

where $C_v = 3R/2$.

$$W_2 = -1.58 \times 10^{-14} \text{ moles} \cdot \left(\frac{3}{2}\right) 8.31 \frac{\text{J}}{\text{mol}\cdot\text{K}} \cdot (18800 - 293) \text{ K}$$

$$W_2 = -3.64 \times 10^{-9} \text{ J}$$

The work done *on* the bubble is then

$$W_2 = 3.64 \times 10^{-9} \text{ J}$$

Process 3: Isochoric cooling

The work done on the bubble during an isochoric process is zero, so $W_3 = 0$ J.

The total work is then the sum of the work on the bubble

$$W_{total} = W_1 + W_2 + W_3$$

$$W_{total} = -2.40 \times 10^{-10} \text{ J} + 3.64 \times 10^{-9} \text{ J} + 0 \text{ J}$$

e) $W_{total} = 3.4 \times 10^{-}9$ J.

Question A2

A thin, uniform rod of length L and mass $M = 0.258$ kg is suspended from a point a distance R away from its center of mass. When the end of the rod is displaced slightly and released it executes simple harmonic oscillation. The period, T, of the oscillation is timed using an electronic timer. The following data is recorded for the period as a function of R. What is the local value of g? Do not assume it is the canonical value of 9.8 m/s^2. What is the length, L, of the rod? No estimation of error in either value is required. The moment of inertia of a rod about its center of mass is $(1/12)ML^2$.

R (m)	T (s)
0.050	3.842
0.075	3.164
0.102	2.747
0.156	2.301
0.198	2.115

R (m)	T (s)
0.211	2.074
0.302	1.905
0.387	1.855
0.451	1.853
0.588	1.900

You *must* show your work to obtain full credit. If you use graphical techniques then you must plot the graph; if you use linear regression techniques then you must show all of the formulae and associated workings used to obtain your result.

Solution

The period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{mgR}} = 2\pi\sqrt{\frac{\frac{1}{12}L^2 + R^2}{gR}}$$

A little math, and

$$g\frac{T^2R}{4\pi^2} - \frac{1}{12}L^2 = R^2.$$

This is of the form

$$mx + b = y$$

if we let

$$y = R^2$$

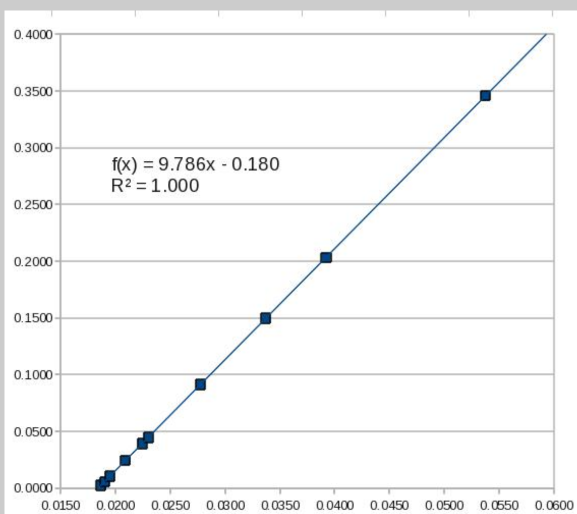
and

$$x = \frac{T^2R}{4\pi^2}$$

Filling out a table of data, we get

R	T	$T^2R/4\pi^2$	R^2
0.050	3.842	0.0187	0.0025
0.075	3.164	0.0190	0.0056
0.102	2.747	0.0195	0.0104
0.156	2.301	0.0209	0.0243
0.198	2.115	0.0224	0.0392
0.211	2.074	0.0230	0.0445
0.302	1.905	0.0278	0.0912
0.387	1.855	0.0337	0.1498
0.451	1.853	0.0392	0.2034
0.588	1.900	0.0538	0.3457

The corresponding graph of $T^2/R/4\pi^2$ versus R^2 ought yield a straight line such that the slope is g and the intercept is $-\frac{1}{12}L^2$.



$$g = 9.7923 \text{ m/s}^2$$

and

$$L = 1.470 \text{ m}$$

Question A3

A light bulb has a solid cylindrical filament of length L and radius a, and consumes power P. You are to design a new light bulb, using a cylindrical filament of the same material, operating at the same voltage, and emitting the same spectrum of light, which will consume power nP. What are the length and radius of the new filament? Assume that the temperature of the filament is approximately uniform across its cross-section; the filament doesn't emit light from the ends; and energy loss due to convection is minimal.

Solution

Since the new bulb emits the same spectrum of light, the emitted power is simply proportional to the area:

$$P \propto 2\pi a L$$

$$P \propto aL$$

If the resistivity of the filament is ρ, the resistance is

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi a^2}$$

and therefore the power is also given by

$$P = \frac{V^2}{R} = \frac{V^2 \pi a^2}{\rho L}$$

$$P \propto \frac{a^2}{L}$$

Combining our conditions,

$$a \propto P^{2/3}$$

$$L \propto P^{1/3}$$

So the new filament must have length $n^{2/3}a$ and length $n^{1/3}L$.

Question A4

In this problem we consider a simplified model of the electromagnetic radiation inside a cubical box of side length L. In this model, the electric field has spatial dependence

$$E(x, y, z) = E_0 \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where one corner of the box lies at the origin and the box is aligned with the x, y, and z axes. Let h be Planck's constant, k_B be Boltzmann's constant, and c be the speed of light.

a. The electric field must be zero everywhere at the sides of the box. What condition does this impose on k_x, k_y, and k_z? (Assume that any of these may be negative, and include cases where one or more of the k_i is zero, even though this causes E to be zero.)

b. In the model, each permitted value of the triple (k_x, k_y, k_z) corresponds to a quantum state. These states can be visualized in a *state space*, which is a notional three-dimensional space with axes corresponding to k_x, k_y, and k_z. How many states occupy a volume s of state space, if s is large enough that the discreteness of the states can be ignored?

c. Each quantum state, in turn, may be *occupied* by photons with frequency $\omega = \frac{f}{2\pi} = c|\mathbf{k}|$, where

$$|\mathbf{k}| = \sqrt{{k_x}^2 + {k_y}^2 + {k_z}^2}$$

In the model, if the temperature inside the box is T, no photon may have energy greater than k_BT. What is the shape of the region in state space corresponding to occupied states?

d. As a final approximation, assume that each occupied state contains exactly one photon. What is the total energy of the photons in the box, in terms of h, k_B, c, T, and the volume of the box V? Again, assume that the temperature is high enough that there are a very large number of occupied states. (*Hint*: divide state space into thin regions corresponding to photons of the same energy.)

Note that while many details of this model are extremely inaccurate, the final result is correct except for a numerical factor.

Solution

a. We require that $\sin(k_xL) = 0$, so that

$$k_xL = n_x\pi$$

for any integer n_x, and similarly for k_y and k_z.

b. The occupied states are equally spaced a distance $\frac{\pi}{L}$ apart. Each can therefore be thought of as taking up volume $\frac{\pi^3}{L^3}$, and the number of states in the volume s is

$$\frac{L^3}{\pi^3}s$$

c. A photon's energy is $E = \hbar\omega = \hbar c|\mathbf{k}|$, where $\hbar = \frac{h}{2\pi}$. Thus the occupied states obey

$$\hbar c|\mathbf{k}| \le k_BT$$

$$|\mathbf{k}| \le \frac{k_BT}{\hbar c}$$

This corresponds to a ball of radius $\frac{k_BT}{\hbar c}$ in state space.

d. As we have seen, the energy of a photon is proportional to its distance $|\mathbf{k}|$ from the origin in state space. Thus consider the spherical shell in state space between radius k and radius $k + dk$. The volume of this region is

$$ds = 4\pi k^2 dk$$

Each photon in the region has energy $\hbar ck$, and from above there are $\frac{L^3}{\pi^3}ds$ photons in the region. Therefore the photons in the region have total energy

$$dE = \hbar ck \cdot \frac{L^3}{\pi^3} \cdot 4\pi k^2 dk$$

$$dE = \frac{4}{\pi^2}\hbar c L^3 k^3 \; dk$$

From above, k ranges from zero to $k_{max} = \frac{k_B T}{\hbar c}$, so the total energy is

$$E = \int_0^{k_{max}} \frac{4}{\pi^2}\hbar c L^3 k^3 \; dk$$

$$E = \frac{4}{\pi^2}\hbar c L^3 \cdot \frac{1}{4}\left(\frac{k_B T}{\hbar c}\right)^4$$

Since the volume of the box is $V = L^3$, and $h = 2\pi\hbar$, this cleans up to

$$E = \frac{8\pi {k_B}^4}{h^3 c^3} T^4 V$$

STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

An AC power line cable transmits electrical power using a sinusoidal waveform with frequency 60 Hz. The load receives an RMS voltage of 500 kV and requires 1000 MW of average power. For this problem, consider only the cable carrying current in one of the two directions, and ignore effects due to capacitance or inductance between the cable and with the ground.

a. Suppose that the load on the power line cable is a residential area that behaves like a pure resistor.

   i. What is the RMS current carried in the cable?

   ii. The cable has diameter 3 cm, is 500 km long, and is made of aluminum with resistivity $2.8 \times 10^{-8}\ \Omega \cdot \mathrm{m}$. How much power is lost in the wire?

b. A local rancher thinks he might be able to extract electrical power from the cable using electromagnetic induction. The rancher constructs a rectangular loop of length a and width $b < a$, consisting of N turns of wire. One edge of the loop is to be placed on the ground; the wire is straight and runs parallel to the ground at a height h much less than the length of the wire. Write the current in the wire as $I = I_0 \sin \omega t$, and assume the return wire is far away.

   i. Determine an expression for the magnitude of the magnetic field at a distance r from the power line cable in terms of I, r, and fundamental constants.

   ii. Where should the loop be placed, and how should it be oriented, to maximize the induced emf in the loop?

   iii. Assuming the loop is placed in this way, determine an expression for the emf induced in the loop (as a function of time) in terms of any or all of I_0, h, a, b, N, ω, t, and fundamental constants.

   iv. Suppose that $a = 5$ m, $b = 2$ m, and $h = 100$ m. How many turns of wire N does the rancher need to generate an RMS emf of 120 V?

c. The load at the end of the power line cable changes to include a manufacturing plant with a large number of electric motors. While the average power consumed remains the same, it now behaves like a resistor in parallel with a 0.25 H inductor.

   i. Does the power lost in the power line cable increase, decrease, or stay the same? (You need not calculate the new value explicitly, but you should show some work to defend your answer.)

   ii. The power company wishes to make the load behave as it originally did by installing a capacitor in parallel with the load. What should be its capacitance?

Solution

a. i. Because the load is purely resistive, the average power is simply

$$P_{av} = V_{rms} I_{rms}$$

so

$$I_{rms} = \frac{P_{av}}{V_{rms}} = 2000 \text{ A}$$

ii. The cross-sectional area of the wire is $A = \pi r^2 = 7.07 \times 10^{-4}$ m^2, so its resistance is

$$R = \frac{\rho L}{A} = 19.8\ \Omega$$

The power loss is then

$$P = I^2 R = 79.2 \text{ MW}$$

b. i. The field is perpendicular to the wire and to the radius, and from Ampere's Law

$$\oint \mathbf{B} \cdot \mathbf{ds} = \mu_0 I_{encl}$$

$$B \cdot 2\pi r = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi r}$$

ii. The induced emf is proportional to the rate of change of the flux through the loop. Since the time dependence of the magnetic field is uniform across space, the rate of change of flux is maximized by maximizing the flux itself. This in turn can be accomplished by maximizing the field in the loop and ensuring that it is normal to the loop. Because the field gets stronger closer to the wire, the loop should be directly below the wire, and since the field is horizontal and perpendicular to the wire at this location, the loop should be vertical and parallel to the wire. Finally, again because the field gets stronger closer to the wire, the long edge of the loop should be vertical.

In summary, the loop should be placed vertically, parallel to the wire and directly beneath it, with the long edge vertical.

iii. From Faraday's law,

$$\mathcal{E} = N \frac{d}{dt} \Phi_B$$

where we have dropped the sign and Φ_B is the magnetic flux through a single loop. The flux, in turn, is defined as

$$\Phi_B = \int \mathbf{B} \cdot \mathbf{dA}$$

Dividing the loop into strips of radial width dr and length b,

$$\Phi_B = \int_{h-a}^{h} B(r) b\, dr$$

$$\Phi_B = \int_{h-a}^{h} \frac{\mu_0 I}{2\pi r} b\, dr$$

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \ln \frac{h}{h-a}$$

So,

$$\mathcal{E} = N \frac{d}{dt} \frac{\mu_0 I b}{2\pi} \ln \frac{h}{h-a}$$

Since the loop is stationary, only I depends on t, and

$$\mathcal{E} = N\frac{\mu_0 b}{2\pi}\ln\frac{h}{h-a}\frac{dI}{dt}$$

$$\mathcal{E} = N\frac{\mu_0 b}{2\pi}\ln\frac{h}{h-a}I_0\omega\cos\omega t$$

iv. Note that the RMS value of $I_0\cos\omega t$ is the same as the RMS value of $I_0\sin\omega t$, *i.e.* I_{rms}. So, taking the RMS value of both sides of our previous result,

$$\mathcal{E}_{rms} = N\frac{\mu_0 b}{2\pi}\ln\frac{h}{h-a}\omega I_{rms}$$

And (conveniently) the frequency $f = \frac{\omega}{2\pi}$, so

$$\mathcal{E}_{rms} = N\mu_0 bf\ln\frac{h}{h-a}I_{rms}$$

With the given numbers,

$$\mu_0 bf\ln\frac{h}{h-a}I_{rms} = 0.0155\text{ V}$$

so that the required number of turns is

$$N = 7757$$

c. i. The inductor adds a new component of the current in the wire out of phase with the voltage; this component does not transmit power, so the in-phase component must remain unchanged. The total current is thus increased, and with it the power lost in the wire increases as well.

ii. The resonant frequency of an LC circuit is given by

$$\omega = \frac{1}{\sqrt{LC}}$$

so that

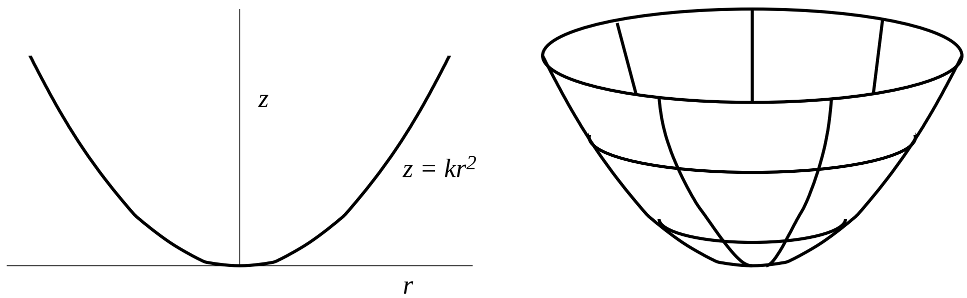
$$C = \frac{1}{\omega^2 L}$$

Here $\omega = 2\pi f = 377\text{ s}^{-1}$, so

$$C = 28.1\ \mu\text{F}$$

Question B2

A particle is constrained to move on the inner surface of a frictionless parabolic bowl whose cross-section has equation $z = kr^2$. The particle begins at a height z_0 above the bottom of the bowl with a horizontal velocity v_0 along the surface of the bowl. The acceleration due to gravity is g.



a. For a particular value of horizontal velocity v_0, which we will name v_h, the particle moves in a horizontal circle. What is v_h in terms of g, z_0, and/or k?

b. Suppose that the initial horizontal velocity is now $v_0 > v_h$. What is the maximum height reached by the particle, in terms of v_0, z_0, g and/or k?

c. Suppose that the particle now begins at a height z_0 above the bottom of the bowl with an initial velocity $v_0 = 0$.

   i. Assuming that z_0 is small enough so that the motion can be approximated as simple harmonic, find the period of the motion in terms any or all of the mass of the particle m, g, z_0, and/or k.

   ii. Assuming that z_0 is not small, will the actual period of motion be greater than, less than, or equal to your simple harmonic approximation above? (You need not calculate the new value explicitly, but you should show some work to defend your answer.)

Solution

a. Let the particle have mass m, let the radius of the bowl at height z_0 be r_0, and let the angle made by the bowl's surface to the horizontal at that height be θ.

   Two forces act on the particle: the normal force and gravity. If the particle moves in a horizontal circle the horizontal component of the net force must equal the centripetal force $F_c = \frac{m{v_h}^2}{r_0}$, whereas the vertical component must be zero. From the free body diagram [Diagram], these conditions are

$$F_N \sin\theta = \frac{m{v_h}^2}{r_0}$$

$$F_N \cos\theta - mg = 0$$

   Combining these,

$$\tan\theta = \frac{{v_h}^2}{g r_0}$$

However, $\tan\theta$ is simply the slope of the bowl $\frac{dz}{dr} = 2kr_0$, so that

$$2ar_0 = \frac{{v_h}^2}{gr_0}$$

Using the fact that $z_0 = k{r_0}^2$,

$$v_h = \sqrt{2gz_0}$$

b. Let the maximum height be z, let the radius of the bowl at this point be r, and let the speed of the particle at this point be v. From conservation of energy,

$$\frac{1}{2}m{v_0}^2 + mgz_0 = \frac{1}{2}mv^2 + mgz$$

Meanwhile, the two forces acting on the particle never exert a torque in the direction of the bowl's axis, and so angular momentum about this axis is conserved. Furthermore, at the point of maximum height the velocity of the particle is entirely tangential to the axis, so the conservation condition is simply

$$mv_0r_0 = mvr$$
$$v = v_0\frac{r_0}{r}$$

or, since $z = kr^2$ and $z_0 = k{r_0}^2$,

$$v = v_0\sqrt{\frac{z_0}{z}}$$

Combining our results,

$$\frac{1}{2}m{v_0}^2 + mgz_0 = \frac{1}{2}m{v_0}^2\frac{z_0}{z} + mgz$$
$$z^2 - \left(\frac{{v_0}^2}{2g} + z_0\right)z + \frac{{v_0}^2}{2g}z_0 = 0$$
$$(z - z_0)\left(z - \frac{{v_0}^2}{2g}\right) = 0$$

The root $z = z_0$ corresponds to our starting condition, so the desired root is

$$z = \frac{{v_0}^2}{2g}$$

Note that we recover $z = z_0$ if $v_0 = v_h$ as we would expect; indeed, the analysis in this section is an alternative path to the previous result.

c. i. We present a force-based approach and an energy-based approach. In each case, let r be the radial position of the particle, so that $z = kr^2$ is the height of the particle above the bottom of the bowl.

The force-based approach begins with the free body diagram. [Diagram] Again, let the angle of the bowl's surface to the horizontal be θ. Because z_0 is small,

$$\sin\theta \approx \theta \approx \tan\theta = \frac{dz}{dr} = 2kr$$

and $\cos\theta \approx 1$.

We can consider the component of force tangential to the bowl, which is $mg\sin\theta$; Newton's third law then gives for the magnitude of the acceleration a

$$ma = mg\sin\theta$$

noting that the acceleration is entirely tangential because the particle is constrained to the surface of the bowl (and there is no centripetal force anymore). The radial acceleration a_r is given by

$$a_r = -a\cos\theta$$

where we have introduced the appropriate sign to match the sign of r. Thus

$$a_r = -g\cos\theta\sin\theta$$

So in the small-z approximation,

$$a_r \approx -g\tan\theta$$

$$a_r = \frac{d^2r}{dt^2} = -2krg$$

The energy-based approach begins with the total energy

$$E = \frac{1}{2}mv^2 + mgz$$

The velocity v is given by

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

Because z is small, $\frac{dz}{dt} \ll \frac{dr}{dt}$, and we conclude that

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + mgkr^2$$

From conservation of energy, $\frac{dE}{dt} = 0$:

$$0 = m\frac{dr}{dt}\frac{d^2r}{dt^2} + 2mgkr\frac{dr}{dt}$$

$$0 = \frac{d^2r}{dt^2} + 2krg$$

Both approaches lead to the standard SHM differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

with angular frequency $\omega = \sqrt{2kg}$; since the period $T = \frac{2\pi}{\omega}$,

$$T = \frac{2\pi}{\sqrt{2kg}}$$

ii. The period is greater than the simple harmonic period. We can see this using both approaches:

In the force-based approach, we obtained the exact equation

$$a_r = -g \cos\theta \sin\theta$$

and approximated it as

$$a_r = -g \tan\theta$$

Since $\cos\theta \sin\theta < \tan\theta$, the exact radial acceleration is smaller than the approximate one, so that the particle takes longer to reach the origin in reality than it does in the approximation, meaning that the period is larger.

In the energy-based approach, we dropped a (positive) term in the formula for the speed v as expressed in terms of $\frac{dr}{dt}$. Therefore we overestimated $\frac{dr}{dt}$, and again the particle takes longer to reach the origin in reality than it does in the approximation.