AAPT AIP UNITED STATES PHYSICS TEAM 2017

USA Physics Olympiad Exam

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Important Instructions for the Exam Supervisor

- This examination consists of two parts: Part A has four questions and is allowed 90 minutes; Part B has two questions and is allowed 90 minutes.
- The first page that follows is a cover sheet. Examinees may keep the cover sheet for both parts of the exam.
- The parts are then identified by the center header on each page. Examinees are only allowed to do one part at a time, and may not work on other parts, even if they have time remaining.
- Allow 90 minutes to complete Part A. Do not let students look at Part B. Collect the answers to Part A before allowing the examinee to begin Part B. Examinees are allowed a 10 to 15 minutes break between parts A and B.
- Allow 90 minutes to complete Part B. Do not let students go back to Part A.
- Ideally the test supervisor will divide the question paper into 3 parts: the cover sheet (page 2), Part A (pages 3-14), Part B (pages 16-25). Examinees should be provided parts A and B individually, although they may keep the cover sheet. The answer sheets should be printed single sided!
- The supervisor *must* collect all examination questions, including the cover sheet, at the end of the exam, as well as any scratch paper used by the examinees. Examinees may *not* take the exam questions. The examination questions may be returned to the students after April 15, 2017.
- Examinees are allowed calculators, but they may not use symbolic math, programming, or graphic features of these calculators. Calculators may not be shared and their memory must be cleared of data and programs. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. Examinees may not use any tables, books, or collections of formulas.

AAPT UNITED STATES PHYSICS TEAM
AIP 2017

USA Physics Olympiad Exam

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Work Part A first. You have 90 minutes to complete all four problems. Each question is worth 25 points. Do not look at Part B during this time.
- After you have completed Part A you may take a break.
- Then work Part B. You have 90 minutes to complete both problems. Each question is worth 50 points. Do not look at Part A during this time.
- Show all your work. Partial credit will be given. Do not write on the back of any page. Do not write anything that you wish graded on the question sheets.
- Start each question on a new sheet of paper. Put your proctor's AAPT ID, your AAPT ID, your name, the question number and the page number/total pages for this problem, in the upper right hand corner of each page. For example,

  Doe, Jamie
  student AAPT ID #
  proctor AAPT ID #
  A1 - 1/3

- A hand-held calculator may be used. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones, PDA's or cameras may not be used during the exam or while the exam papers are present. You may not use any tables, books, or collections of formulas.
- Questions with the same point value are not necessarily of the same difficulty.
- **In order to maintain exam security, do not communicate any information about the questions (or their answers/solutions) on this contest until after April 8, 2017.**

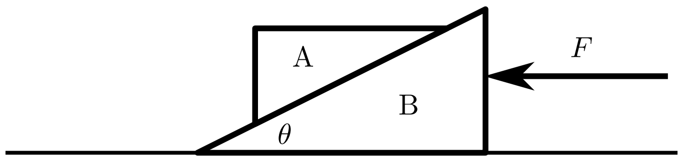
Possibly Useful Information. You may use this sheet for both parts of the exam.

$g = 9.8$ N/kg	$G = 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2
$k = 1/4\pi\epsilon_0 = 8.99 \times 10^9$ N $\cdot$ m^2/C^2	$k_{\mathrm{m}} = \mu_0/4\pi = 10^{-7}$ T $\cdot$ m/A
$c = 3.00 \times 10^8$ m/s	$k_{\mathrm{B}} = 1.38 \times 10^{-23}$ J/K
$N_{\mathrm{A}} = 6.02 \times 10^{23}$ (mol)$^{-1}$	$R = N_{\mathrm{A}} k_{\mathrm{B}} = 8.31$ J/(mol $\cdot$ K)
$\sigma = 5.67 \times 10^{-8}$ J/(s $\cdot$ m^2 $\cdot$ K^4)	$e = 1.602 \times 10^{-19}$ C
1 eV $= 1.602 \times 10^{-19}$ J	$h = 6.63 \times 10^{-34}$ J $\cdot$ s $= 4.14 \times 10^{-15}$ eV $\cdot$ s
$m_e = 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2	$(1+x)^n \approx 1 + nx$ for $\lvert x \rvert \ll 1$
$m_p = 1.673 \times 10^{-27}$ kg $= 938$ MeV/c^2	$\ln(1+x) \approx x$ for $\lvert x \rvert \ll 1$
$\sin\theta \approx \theta - \frac{1}{6}\theta^3$ for $\lvert\theta\rvert \ll 1$	$\cos\theta \approx 1 - \frac{1}{2}\theta^2$ for $\lvert\theta\rvert \ll 1$

Part A

Question A1

A pair of wedges are located on a horizontal surface. The coefficient of friction (both sliding and static) between the wedges is μ, the coefficient of friction between the bottom wedge B and the horizontal surface is μ, and the angle of the wedge is θ. The mass of the top wedge A is m, and the mass of the bottom wedge B is $M = 2m$. A horizontal force F directed to the left is applied to the bottom wedge as shown in the figure.



Determine the range of values for F so that the the top wedge does not slip on the bottom wedge. Express your answer(s) in terms of any or all of m, g, θ, and μ.

Solution

Wedge: General Solution with mass ratio α
Note that the problem was changed so that $\alpha = 2$.
Solution 1
For the entire system,

$$F - \mu(1+\alpha)mg = (1+\alpha)ma \Rightarrow ma = \frac{F}{1+\alpha} - \mu mg$$

Critical condition 1: When F, and subsequently a, is small, the top wedge tends down and static friction is **UP** the ramp.

For the top wedge:

$$\perp \text{ to ramp}: \quad N\cos\theta + f\sin\theta = mg \Rightarrow N = \frac{mg}{\cos\theta} - f\tan\theta \qquad \text{(A1-1)}$$

$$\Rightarrow \quad f \le \mu\frac{mg}{\cos\theta} - f\tan\theta \qquad \text{(A1-2)}$$

$$\Rightarrow \quad f \le \frac{\mu mg}{\cos\theta + \mu\sin\theta} \qquad \text{(A1-3)}$$

$$\parallel \text{ to ramp}: \quad N\sin\theta - f\cos\theta = ma = \frac{F}{1+\alpha} - \mu mg \qquad \text{(A1-4)}$$

Rearrange the last equation, one arrives at:

$$F \ge (1+\alpha)mg\frac{\sin\theta(1+\mu^2)}{\cos\theta + \mu\sin\theta} = (1+\alpha)mg\frac{(1+\mu^2)\tan\theta}{1+\mu\tan\theta} \equiv F_{\min}$$

Critical condition 2: When F is large, the top wedge tends up and static friction is **DOWN** the ramp.

$$\perp \text{ to ramp}: \quad N\cos\theta = f\sin\theta + mg \Rightarrow N = \frac{mg}{\cos\theta} + f\tan\theta \tag{A1-5}$$

$$\Rightarrow f \le \frac{\mu mg}{\cos\theta - \mu\sin\theta} \tag{A1-6}$$

$$\parallel \text{ to ramp}: \quad N\sin\theta + f\cos\theta = ma = \frac{F}{1+\alpha} - \mu mg \tag{A1-7}$$

Again some rearranging late,

$$F \le (1+\alpha)mg\frac{2\mu + (1-\mu^2)\tan\theta}{1-\mu\tan\theta} \equiv F_{\text{max}}$$

Carefully examining the results, one should note that:

- When there is no relative motion between the wedges and the floor, $a = 0$ and $F = f_{\text{static}} \le \mu(1+\alpha)mg$. In this case, the condition that the top does not slip is:

  $$f_{\text{static}} = \mu mg\cos\theta \ge mg\sin\theta.$$

  Therefore the range of F is $[0, \mu(1+\alpha)mg]$

- Examining the express of F_{max} above, one also notes that if $\mu > \cot\theta$, F can be of any value and the top wedge still does not slip. Hence $F_{\text{max}} = \infty$ if $\mu > \cot\theta$.

Solution 2

The no slip condition is given by

$$\tan\phi \le \mu$$

where ϕ is the angle between the plane and the vertical, and μ the coefficient of friction.

We can move into a virtual vertical, however, by recognizing that "down" will make an effective angle of β with the vertical according to

$$\tan\beta = \frac{a}{g},$$

where a is the horizontal acceleration of the blocks, which must be given by

$$a = \frac{F - \mu mg(1+\alpha)}{m(1+\alpha)}$$

or

$$F = m(a + \mu g)(1+\alpha)$$

So the top block will not slip so long as

$$|\theta - \beta| \le \phi$$

Consider the minimum a_{min}, which happens when

$$\beta = \theta - \phi.$$

Take the tangent of both sides, and

$$\frac{a_{\text{min}}}{g} = \frac{\tan\theta - \mu}{1 + \mu\tan\theta}$$

This is only meaningful for $\tan\theta > \mu$, otherwise the answer is simply $a_{\text{min}} = 0$.

Consider the minimum a_{max}, which happens when

$$\beta = \theta + \phi$$

Take the tangent of both sides, and

$$\frac{a_{\text{max}}}{g} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

This is only meaningful for $\cot\theta > \mu$, otherwise the answer is simply $a_{\text{max}} = \infty$.

In short, we toss out the negative answers for a.

Question A2

Consider two objects with equal heat capacities C and initial temperatures T_1 and T_2. A Carnot engine is run using these objects as its hot and cold reservoirs until they are at equal temperatures. Assume that the temperature changes of both the hot and cold reservoirs is very small compared to the temperature during any one cycle of the Carnot engine.

a. Find the final temperature T_f of the two objects, and the total work W done by the engine.

Solution

Since a Carnot engine is reversible, it produces no entropy,

$$dS_1 + dS_2 = \frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = 0.$$

By the definition of heat capacity, $dQ_i = C\,dT_i$, so

$$\frac{dT_1}{T_1} = -\frac{dT_2}{T_2}.$$

Integrating this equation shows that T_1T_2 is constant, so the final temperature is

$$T_f = \sqrt{T_1T_2}.$$

The change in thermal energy of the objects is

$$C(T_f - T_1) + C(T_f - T_2) = C\left[2\sqrt{T_1T_2} - T_1 - T_2\right].$$

By the First Law of Thermodynamics, the missing energy has been used to do work, so

$$W = C\left[T_1 + T_2 - 2\sqrt{T_1T_2}\right].$$

Now consider three objects with equal and constant heat capacity at initial temperatures $T_1 = 100$ K, $T_2 = 300$ K, and $T_3 = 300$ K. Suppose we wish to raise the temperature of the third object.

To do this, we could run a Carnot engine between the first and second objects, extracting work W. This work can then be dissipated as heat to raise the temperature of the third object. Even better, it can be stored and used to run a Carnot engine between the first and third object in reverse, which pumps heat into the third object.

Assume that all work produced by running engines can be stored and used without dissipation.

b. Find the minimum temperature T_L to which the first object can be lowered.

Solution

By the Second Law of Thermodynamics, we must have $T_L = 100\text{K}$. Indeed, if $T_L < 100\text{K}$, then it would be possible to extract work by restoring the objects to their original temperatures.

c. Find the maximum temperature T_H to which the third object can be raised.

Solution

The entropy of an object with constant heat capacity is

$$S = \int \frac{dQ}{T} = C \int \frac{dT}{T} = C \ln T.$$

Since the total entropy remains constant, $T_1 T_2 T_3$ is constant; this is a direct generalization of the result for T_f found in part (a). Energy is also conserved, as it makes no sense to leave stored energy unused, so $T_1 + T_2 + T_3$ is constant.

When one object is at temperature T_H, the other two must be at the same lower temperature T_0, or else further work could be extracted from their temperature difference, so

$$T_1 + T_2 + T_3 = T_H + 2T_0, \quad T_1 T_2 T_3 = T_H T_0^2.$$

Plugging in temperatures with values divided by 100 for convenience, and eliminating T_0 gives

$$T_H (7 - T_H)^2 = 36.$$

We know that $T_H = 1$ is one (spurious) solution, since this is the minimum possible final temperature as found in part (b). The other roots are $T_H = 4$ and $T_H = 9$ by the quadratic formula. The solution $T_H = 9$ is impossible by energy conservation, so

$$T_H = 400\text{K}.$$

It is also possible to solve the problem more explicitly. For example, one can run a Carnot cycle between the first two objects until they are at the same temperature, then run a Carnot cycle in reverse between the last two objects using the stored work. At this point, the first two objects will no longer be at the same temperature, so we can repeat the procedure; this yields an infinite series for T_H. Taking only the first term of this series yields a slightly worse result, $T_H \approx 395\text{K}$.

Another explicit method is to continuously switch between running one Carnot engine forward and another Carnot engine in reverse; this yields three differential equations for T_1, T_2, and T_3. Solving the equations and setting $T_1 = T_2$ yields $T_3 = T_H$.

Question A3

A ship can be thought of as a symmetric arrangement of soft iron. In the presence of an external magnetic field, the soft iron will become magnetized, creating a second, weaker magnetic field. We want to examine the effect of the ship's field on the ship's compass, which will be located in the middle of the ship.

Let the strength of the Earth's magnetic field near the ship be B_e, and the orientation of the field be horizontal, pointing directly toward true north.

The Earth's magnetic field B_e will magnetize the ship, which will then create a second magnetic field B_s in the vicinity of the ship's compass given by

$$\vec{\mathbf{B}}_s = B_e\left(-K_b \cos\theta\, \hat{\mathbf{b}} + K_s \sin\theta\, \hat{\mathbf{s}}\right)$$

where K_b and K_s are positive constants, θ is the angle between the heading of the ship and magnetic north, measured clockwise, $\hat{\mathbf{b}}$ and $\hat{\mathbf{s}}$ are unit vectors pointing in the forward direction of the ship (bow) and directly right of the forward direction (starboard), respectively.

Because of the ship's magnetic field, the ship's compass will no longer necessarily point North.

a. Derive an expression for the deviation of the compass, $\delta\theta$, from north as a function of K_b, K_s, and θ.

Solution

Add the fields to get the local field. The northward component is

$$B_{north} = B_e - B_e K_b \cos\theta\cos\theta - B_e K_s \sin\theta\sin\theta$$

while the eastward component is

$$B_{east} = -B_e K_b \sin\theta\cos\theta + B_e K_s \cos\theta\sin\theta$$

The deviation is given by

$$\tan\delta\theta = (K_s - K_b)\frac{\sin\theta\cos\theta}{1 - K_b\cos^2\theta - K_s\sin^2\theta}$$

In general, K_b and K_s are small enough to ignore in the denominator.

b. Assuming that K_b and K_s are both much smaller than one, at what heading(s) θ will the deviation $\delta\theta$ be largest?

Solution

By inspection, $\theta = 45^\circ$ will yield the largest deviation.

A pair of iron balls placed in the same horizontal plane as the compass but a distance d away can be used to help correct for the error caused by the induced magnetism of the ship.

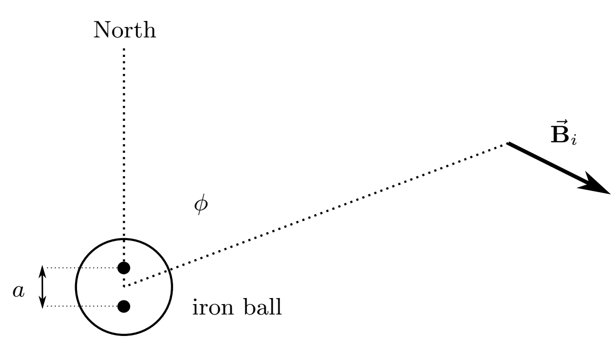
A binnacle, protecting the ship's compass in the center, with two soft iron spheres to help correct for errors in the compass heading. The use of the spheres was suggested by Lord Kelvin.

Just like the ship, the iron balls will become magnetic because of the Earth's field B_e. As spheres, the balls will individually act like dipoles. A dipole can be thought of as the field produced by two magnetic monopoles of strength $\pm m$ at two different points.

The magnetic field of a single pole is

$$\vec{\mathbf{B}} = \pm m \frac{\hat{\mathbf{r}}}{r^2}$$

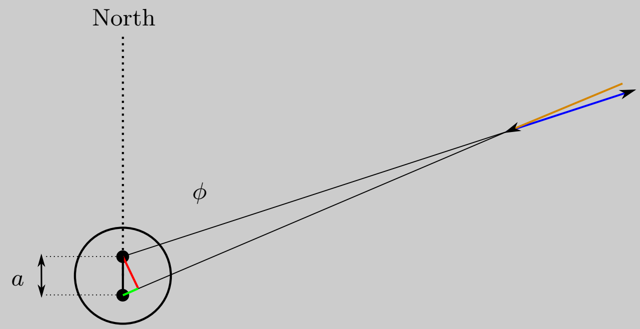
where the positive sign is for a north pole and the negative for a south pole. The dipole magnetic field is the sum of the two fields: a north pole at $y = +a/2$ and a south pole at $y = -a/2$, where the y axis is horizontal and pointing north. a is a small distance much smaller than the radius of the iron balls; in general $a = K_i B_e$ where K_i is a constant that depends on the size of the iron sphere.



c. Derive an expression for the magnetic field $\vec{\mathbf{B}}_i$ from the iron a distance $d \gg a$ from the center of the ball. Note that there will be a component directed radially away from the ball and a component directed tangent to a circle of radius d around the ball, so using polar coordinates is recommended.

Solution

This problem is not near as difficult as it looks.



Consider first the diagram above, with a focus on the black red green triangle. The black side has length a. The angle between the green and black sides is ϕ, so the length of the red side is $a \sin\phi$ and the length of the green side is $a \cos\phi$.

The magnetic field strength from one magnetic pole a distance d away is given by

$$B = \pm m \frac{1}{d^2}$$

The sum of the two fields has two components. The angular component is a measure of the "opening" of the triangle formed by the two vectors, and since the two vectors basically have the same length, we can use similar triangles to conclude

$$\frac{a \sin\phi}{d} \approx \frac{B_\phi}{B}$$

or

$$B_\phi = m \frac{a}{d^3} \sin\phi = B_e \frac{mK_i}{d^3} \sin\phi$$

As expected, this component vanishes for $\phi = 0$.

The radial component is given by the difference in the lengths of the two field vectors, or

$$B_r = m\left(\frac{1}{d^2} - \frac{1}{(d+x)^2}\right) = m\frac{1}{d^2}\left(1 - \frac{1}{(1+x/d)^2}\right) \approx m\frac{1}{d^2}\frac{2x}{d},$$

where $x = a\cos\phi$ is the green length, so

$$B_r = B_e \frac{mK_i}{d^3} 2\cos\phi$$

That wasn't so bad, was it?

d. If placed directly to the right and left of the ship compass, the iron balls can be located at a distance d to cancel out the error in the magnetic heading for any angle(s) where $\delta\theta$ is largest. Assuming that this is done, find the resulting expression for the combined deviation $\delta\theta$ due to the ship and the balls for the magnetic heading for all angles θ.

Solution

Note that the two iron balls created a magnetic field near the compass that behaves in a similar fashion to the ship as a whole. There is a component directed toward the bow given by

$$B_b = -2B_\theta \propto \sin\phi \propto \cos\theta$$

and a component directed toward the starboard given by

$$B_s = 2B_r \propto \cos\phi \propto \sin\theta$$

where the factors of 2 are because there are two iron balls. Note that θ is the ship heading while ϕ is the angle between north and the location of the compass relative to (one of) the ball(s).

If it is corrected for the maximum angles it will necessarily cancel out the induced ship field for all of the angles, so that

$$\delta\theta = 0$$

for all θ. Effectively, this means placing the balls to make $K_b = K_s$.

Question A4

Relativistic particles obey the mass energy relation

$$E^2 = (pc)^2 + (mc^2)^2$$

where E is the relativistic energy of the particle, p is the relativistic momentum, m is the mass, and c is the speed of light.

A proton with mass m_p and energy E_p collides head on with a photon which is massless and has energy E_b. The two combine and form a new particle with mass m_Δ called Δ, or "delta". It is a one dimensional collision that conserves both relativistic energy and relativistic momentum.

a. Determine E_p in terms of m_p, m_Δ, and E_b. You may assume that E_b is small.

Solution

This problem will be solved in units of $c = 1$, in order to make the math easier.

First, the exact solution, which you didn't need to do.

The easiest approach is to transform to an inertial frame where the proton particle is at rest before the collision. Then before the collision we have for the proton

$$E_p = m_p,$$

and for the photon

$$E_\gamma = |p_\gamma|,$$

and for the Δ,

$${E_\Delta}^2 = {p_\Delta}^2 + {m_\Delta}^2,$$

By energy conservation,

$$E_p + E_\gamma = E_\Delta.$$

By momentum conservation we have

$$p_\gamma = p_\Delta$$

Combine the above, and

$$(m_p + E_\gamma)^2 = {E_\gamma}^2 + {m_\Delta}^2,$$

or

$$E_\gamma = \frac{{m_\Delta}^2 - {m_p}^2}{2m_p}$$

But this is really E_γ, the energy of the photon in our special frame where the proton is at rest. So we want to boost to a frame where the energy of the photon is E_b. We can use the Lorentz transformation to do this, but it is a little neater to realize that

$$E = hf$$

for photons, and apply the doppler shift. In this case,

$$\frac{E_b}{E_\gamma} = \sqrt{\frac{1-\beta}{1+\beta}}$$

where β is the velocity parameter of the proton in the inertial frame where the photon has energy E_b.

If the energy ratio is α, we have

$$\beta = \frac{1-\alpha^2}{1+\alpha^2}$$

or

$$\beta = \frac{({m_\Delta}^2 - {m_p}^2)^2 - 4{m_p}^2{E_b}^2}{({m_\Delta}^2 - {m_p}^2)^2 + 4{m_p}^2{E_b}^2}$$

The energy of the proton in this frame is given by knowing the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

If you substitute in the right order, this isn't so bad:

$$\gamma = \frac{1+\alpha^2}{2\alpha}$$

and then the proton energy in the correct rest frame is

$$E_p = \gamma m_p = \frac{m_p}{2}\left(\alpha + \frac{1}{\alpha}\right)$$

or

$$E_p = \frac{m_p}{2}\left(\frac{2m_pE_b}{{m_\Delta}^2 - {m_p}^2} + \frac{{m_\Delta}^2 - {m_p}^2}{2m_pE_b}\right)$$

After the problem was written, it was decide to let E_b be a small quantity; mainly because it is. This makes the math easier, but before I do it, notice that the second term is much larger than the first, so

$$E_p \approx \frac{{m_\Delta}^2 - {m_p}^2}{4E_b}$$

Note that the following reuses symbols from above, but does everything in the lab frame, as such, the symbols are *not* necessarily the same as above.

In the lab frame, using the approximation that E_b is small, conserve momentum:

$$p_p - p_b = p_\Delta$$

and energy

$$E_p + E_b = E_\Delta$$

square both expressions, and drop terms with ${E_b}^2$,

$${p_p}^2 - 2p_pp_b = p_\Delta^2$$

and

$${E_p}^2 + 2E_pE_b = {E_\Delta}^2$$

subtract momentum squared expression from energy squared expression,

$${m_p}^2 + 2E_pE_b + 2p_pE_b = {m_\Delta}^2$$

Then

$$E_p + p_p = \frac{{m_\Delta}^2 - {m_p}^2}{2E_b}$$

Since E_b is small, this quantity must be large. But this means ultrarelativistic protons, so $E_p = p_p$.

b. In this case, the photon energy E_b is that of the cosmic background radiation, which is an EM wave with wavelength 1.06 mm. Determine the energy of the photons, writing your answer in electron volts.

Solution

$$E = \frac{hc}{\lambda} = 0.00112 \text{ eV}$$

c. Assuming this value for E_b, what is the energy of the proton, in electron volts, that will allow the above reaction? This sets an upper limit on the energy of cosmic rays. The mass of the proton is given by $m_p c^2 = 938$ MeV and the mass of the Δ is given by $m_\Delta c^2 = 1232$ MeV.

Solution

Put the numbers into the adventure above:

$$E_p = 1.4 \times 10^{20} \text{ eV}$$

The following relationships may be useful in solving this problem:

velocity parameter	$\beta = \frac{v}{c}$
Lorentz factor	$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
relativistic momentum	$p = \gamma\beta mc$
relativistic energy	$E = \gamma mc^2$
relativistic doppler shift	$\frac{f}{f_0} = \sqrt{\frac{1-\beta}{1+\beta}}$

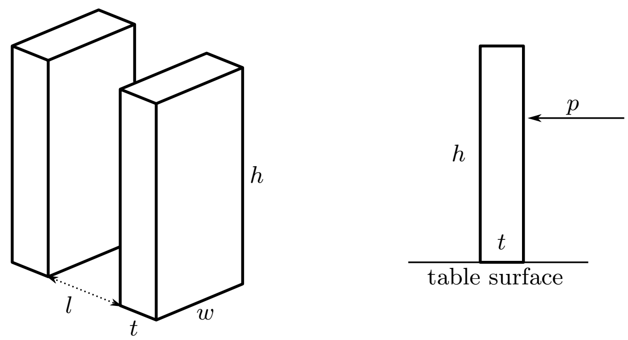
STOP: Do Not Continue to Part B

If there is still time remaining for Part A, you should review your work for Part A, but do not continue to Part B until instructed by your exam supervisor.

Part B

Question B1

Suppose a domino stands upright on a table. It has height h, thickness t, width w (as shown below), and mass m. The domino is free to rotate about its edges, but will not slide across the table.



a. Suppose we give the domino a sharp, horizontal impulsive push with total momentum p.

   i. At what height H above the table is the impulse p required to topple the domino smallest?

   ii. What is the minimum value of p to topple the domino?

Solution

First we'll look for the height H at which we should push to topple the domino with the least momentum. A convenient method is to look at the angular momentum in the domino because this is easy to calculate and describes rotational motion. Because the push is horizontal, the moment arm of the push (about the domino's rotational axis) is purely vertical. That means the angular momentum of the push is $p * H$, where p is the momentum imparted and H is the height of the push. There is some minimum angular momentum L_{min} to topple the domino, so we set $L_{min} = p * H$. The bigger H, the smaller p, so we should choose the largest possible H. In other words, we should push at the very top of the domino, $H = h$. While we're on this part, note that if the push weren't constrained to be horizontal, the push could be a little bit smaller since the moment arm could be the entire diagonal of the thin edge of the domino.

Next we calculate the minimum p using energy. As the domino rotates, it converts kinetic energy to potential energy, so we'll calculate both. The domino's potential energy is greatest when its center of mass is directly over the contact point of the domino and the table. That height is half the diagonal of the domino, so the distance from the contact point to the center of the domino is $\frac{1}{2}\sqrt{t^2 + h^2}$. From the push until it reaches this point, the domino's potential energy increases by

$$\Delta U = \frac{1}{2}mg(\sqrt{t^2 + h^2} - h)$$

. Then the domino topples. By conservation of energy, ΔU is how much rotational kinetic energy the domino must have begun with.

Next we find the kinetic energy using the rotational kinetic energy formula, $T = \frac{1}{2}L^2/I$. The moment of inertia of the domino about its contact point with the table is $I = \frac{1}{3}m(h^2 + t^2)$.

You can find this with an integral, or if you know the moment of inertia about the center ($\frac{1}{12}m(h^2+t^2)$), you can use the parallel axis theorem. We know the momentum $L = ph$, so the kinetic energy is

$$T = \frac{1}{2}\frac{L^2}{I} = \frac{1}{2}\frac{(ph)^2}{(1/3)m(h^2+t^2)} = \frac{3}{2}\frac{p^2h^2}{m(h^2+t^2)}$$

Setting the initial kinetic energy equal to the gain in potential energy and solving for p, we get

$$p_{min} = \frac{1}{\sqrt{3}}\frac{m}{h}\sqrt{g(\sqrt{t^2+h^2}-h)(h^2+t^2)}$$

b. Next, imagine a long row of dominoes with equal spacing l between the nearest sides of any pair of adjacent dominos, as shown above. When a domino topples, it collides with the next domino in the row. Imagine this collision to be completely inelastic. What fraction of the total kinetic energy is lost in the collision of the first domino with the second domino?

Solution

Let's first examine the kinematics. Right after the collision, the dominoes touch at a height $\sqrt{h^2-l^2}$ above the table. The second domino is vertical, while the first is rotated so that the angle between its leading edge and the table is $\theta = \arccos(l/h)$. Let the dominoes' angular velocities be ω_1 and ω_2 respectively. After the dominoes collide, they'll stick together because the collision is assumed to be completely inelastic. If we take the two parts of the dominoes that are in contact, they must have the same horizontal velocity component in order for the dominoes to stay in contact. For the first domino, this velocity is $\omega_1 h \sin\theta = \omega_1\sqrt{h^2-l^2}$. For the second domino it is ω_2 times the height of the impact point, so $\omega_2\sqrt{h^2-l^2}$. Because these velocity components must be the same, we can conclude that $\omega_1 = \omega_2$.

This in turn means that the dominoes have the same angular momentum as each other, measured relative to their respective rotation axes. If we look at the collision, the forces between the dominoes are purely horizontal because the dominoes faces' are frictionless, so they only exert normal forces. The second domino is vertical, so all its normal forces are purely horizontal. These horizontal normal forces trade angular momentum between the two dominoes. They have the same moment arm (IE he height of the collision), so the amount of angular momentum transferred out of the first domino is equal to the amount gained by the second domino, both measured relative to the dominoes' respective rotation axes. Because the dominoes are identical and have the same angular velocity, they have the same angular momentum. This means each domino has half as much angular momentum about its rotation axis as the first domino had just before the collision. (Note that forces from the table cannot change the angular momentum of the dominoes about their respective rotational axes because such forces have zero moment arm, so only the inter-domino forces need to be examined here.)

Kinetic energy scales with the square of angular momentum, so each domino has a quarter as much kinetic energy after the collision as the first domino had before it. That means the total kinetic energy after the collision is half what it was before the collision. The fraction of kinetic energy lost is one half.

c. After the collision, the dominoes rotate in such a way so that they always remain in contact. Assume that there is no friction between the dominoes and the first domino was given the smallest possible push such that it toppled. What is the minimum l such that the second domino will topple?

You may work to lowest nontrivial order in the angles through which the dominoes have rotated. Equivalently, you may approximate $t,\ l \ll h$.

Solution

As in section (a), we will find the highest potential energy of the system and make sure the initial kinetic energy is high enough to get the system to that point.

At any time after the collision, let us call the angle that the first domino has rotated past its point of highest potential energy α, and the angle that the second domino has rotated past its point of highest potential energy β. Also, let's call the angle a domino rotates from its standing position up to its point of highest potential energy ϕ, so the dominoes have rotated $\phi+\alpha$ and $\phi+\beta$ respectively.

The dominoes need to be in contact. The top right corner of the first domino has moved horizontally a distance $h\sin(\phi+\alpha)$ which we will approximate as $h(\phi+\alpha)$ using the small angle approximation. The top left corner of the second domino moves horizontally forward by $h\sin(\phi+\beta)+t(1-\cos(\phi+\beta))$. We ignore the cosine term as second order in the rotation angle and approximate this as $h(\phi+\beta)$.

The y-coordinate of the upper right corner of the first domino is $h(1-\cos(\phi+\alpha))\approx h$. The y-coordinate of the upper left corner of the second domino is $h(1-\cos(\phi+\beta))+t\sin(\phi+\beta)\approx h+t(\phi+\beta)$ There is a first-order difference in y-coordinates of the two corners, but this means the difference in x coordinate between the top left corner of the second domino and the top right corner of the first domino is second-order in the rotation angles. We conclude that to first order

$$h(\phi+\alpha)=l+h(\phi+\beta)$$

because this condition puts the top right corner of the first domino at the same position as the top left of the second domino. We rewrite this as

$$\alpha=\frac{l}{h}+\beta$$

The potential energy of the first domino, setting zero potential energy to be when the domino is upright, is $U_1=\Delta U(1-(\alpha/\phi)^2)$ to second order in α, and for the second domino, $U_2=\Delta U(1-(\beta/\phi)^2)$. These figures come from fitting a quadratic whose peak is when the center of mass is above the rotation point and which is zero when the domino is upright. The total potential energy is U_1+U_2, and using the relation between α and β, it is minimized for

$$\beta_{max}=\frac{-l}{2h}$$

In other words, the second domino is as far away from rotating to the top of arc as the first domino has rotated past the top of its arc. This gives a maximum potential energy

$$U_{max}=2\Delta U\left(1-\frac{l^2}{4t^2}\right)$$

where we have used the approximation $\phi \approx t/h$.

At impact, $U_{impact} \approx \Delta U(-l^2/t^2 + 2l/t)$. Before impact, the kinetic energy is $\Delta U - U_{impact}$ because the maximum potential energy before impact was ΔU, and the kinetic energy was zero there. The kinetic energy just after the collision is then $(\Delta U - U_{impact})/2$. Setting this equal to the potential energy gain as the two dominoes rotate to their highest potential energy, U_{max}, we have

$$\frac{1}{2}(\Delta U - U_{impact}) = U_{max} - U_{impact}$$

or

$$\frac{1}{2}\Delta U = U_{max} - \frac{1}{2}U_{impact}$$

Plugging in the earlier expressions for all these gives

$$\frac{1}{2}\Delta U = 2\Delta U\left(1 - \frac{l^2}{4t^2}\right) - \Delta U\left(\frac{l}{t} - \frac{l^2}{2t^2}\right)$$

and solving this yields

$$l = \frac{3}{2}t$$

to first order in t. So in the limit of tall, thin dominoes, the second domino must be spaced at least $3/2t$ from the first domino in order to topple.

d. A row of toppling dominoes can be considered to have a propagation speed of the length $l + t$ divided by the time between successive collisions. When the first domino is given a minimal push just large enough to topple and start a chain reaction of toppling dominoes, the speed increases with each domino, but approaches an asymptotic speed v.

Suppose there is a row of dominoes on another planet. These dominoes have the same density as the dominoes previously considered, but are twice as tall, wide, and thick, and placed with a spacing of $2l$ between them. If this row of dominoes topples with the same asymptotic speed v previously found, what is the local gravitational acceleration on this planet?

Solution

The speed v can depend on g, h, w, l. To get a quantity with dimensions $[LT^{-1}]$, we must take

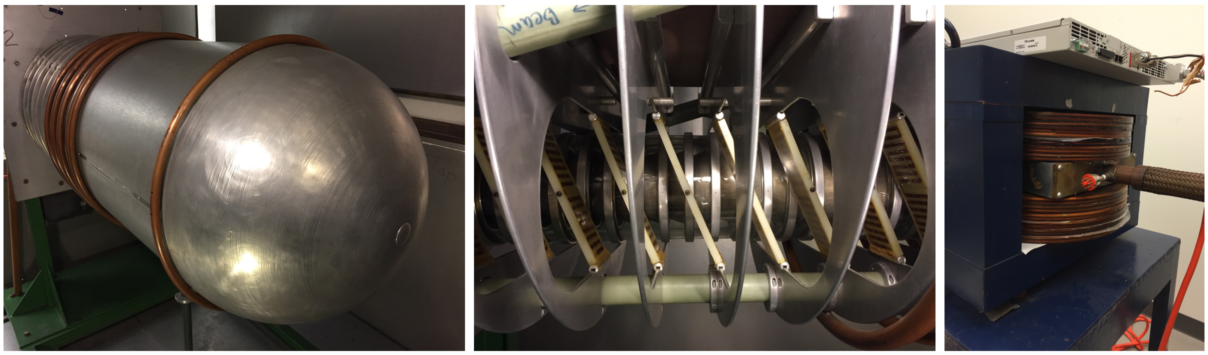
$$v = c\sqrt{gL}$$

where L is some length made from h, w, and l.

On the new planet, v is the same and L is twice as much, no matter what combination of h, w, and l it is. Thus, g must be half as great on the new planet.

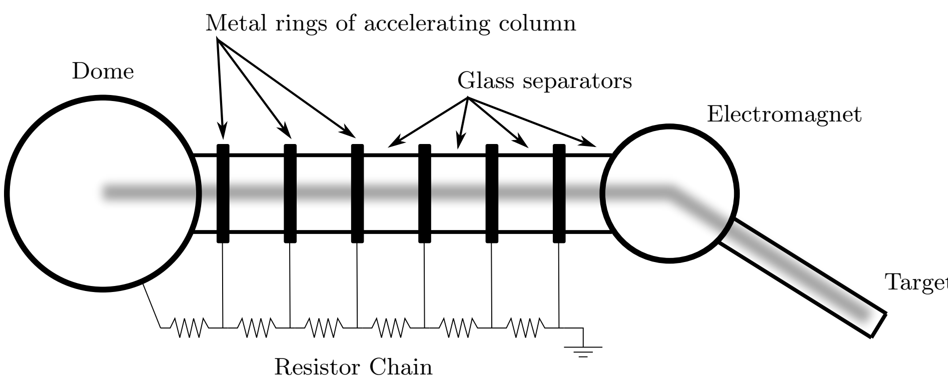
Question B2

Beloit College has a "homemade" 500 kV VanDeGraff proton accelerator, designed and constructed by the students and faculty.



Accelerator dome (assume it is a sphere); accelerating column; bending electromagnet

The accelerator dome, an aluminum sphere of radius $a = 0.50$ meters, is charged by a rubber belt with width $w = 10$ cm that moves with speed $v_b = 20$ m/s. The accelerating column consists of 20 metal rings separated by glass rings; the rings are connected in series with 500 MΩ resistors. The proton beam has a current of 25 μA and is accelerated through 500 kV and then passes through a tuning electromagnet. The electromagnet consists of wound copper pipe as a conductor. The electromagnet effectively creates a uniform field B inside a circular region of radius $b = 10$ cm and zero outside that region.



Only six of the 20 metals rings and resistors are shown in the figure. The fuzzy grey path is the path taken by the protons as they are accelerated from the dome, through the electromagnet, into the target.

a. Assuming the dome is charged to 500 kV, determine the strength of the electric field at the surface of the dome.

Solution

The electric potential is given by

$$V = \frac{q}{4\pi\epsilon_0 a}$$

and the electric field is given by

$$E = \frac{q}{4\pi\epsilon_0 a^2}$$

so

$$E = \frac{V}{a} = 10^6 \text{ V/m}$$

b. Assuming the proton beam is off, determine the time constant for the accelerating dome (the time it takes for the charge on the dome to decrease to $1/e \approx 1/3$ of the initial value.

Solution

The time constant is given by

$$\tau = RC$$

where

$$C = Q/V = 4\pi\epsilon_0 a = 5.56 \times 10^{-11} \text{ F}$$

and

$$R = 20r_0 = 10^{10}\ \Omega$$

so

$$\tau = RC = 0.556 \text{ s}$$

c. Assuming the 25 μA proton beam is on, determine the surface charge density that must be sprayed onto the charging belt in order to maintain a steady charge of 500 kV on the dome.

Solution

There are several ways the dome can discharge, two of which are along the resistors and the proton beam. We will ignore any other path.

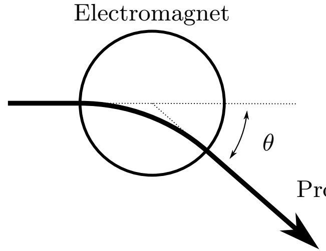
At 500kV, the current through the resistor chain is 50 μA, from $V = IR$. So the total current I needed to be supplied to the dome is 75 μA. This is sprayed onto the belt, which moves at a rate of

$$\frac{\delta A}{\Delta t} = v_b w$$

so the necessary surface charge density is

$$\sigma = \frac{I}{v_b w} = \frac{(75\ \mu\text{C/s})}{(20 \text{ m/s})(0.10 \text{ m})} = 37.5\ \mu\text{C/m}^2$$

d. The proton beam enters the electromagnet and is deflected by an angle $\theta = 10°$. Determine the magnetic field strength.



Solution

Start with

$$F = qvB$$

where F is the force on the protons, and v the velocity.

The protons are non-relativistic, so

$$\frac{1}{2}mv^2 = qV$$

They move in a circle of radius r inside the field, given by

$$\frac{mv^2}{r} = qvB$$

Combine, and

$$r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{q}}$$

We can solve this for the magnetic field strength

$$B = \frac{1}{r}\sqrt{\frac{2mV}{q}}$$

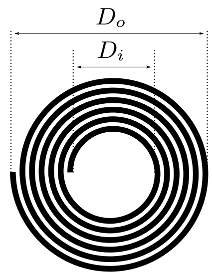
Sketch two circles, one of radius b, the other of radius r, that intersect perpendicular to each other. Half of the deflection angle is then given by

$$\tan\frac{\theta}{2} = \frac{b}{r}$$

Combine, and

$$B = \frac{\tan\theta/2}{b}\sqrt{\frac{2mV}{q}} = 0.0894 \text{ T}$$

e. The electromagnet is composed of layers of spiral wound copper pipe; the pipe has inner diameter $d_i = 0.40$ cm and outer diameter $d_o = 0.50$ cm. The copper pipe is wound into this flat spiral that has an inner diameter $D_i = 20$ cm and outer diameter $D_o = 50$ cm. Assuming the pipe almost touches in the spiral winding, determine the length L in one spiral.



Solution

Treat the problem as two dimensional.

The area of the spiral is

$$A = \frac{\pi}{4}({D_o}^2 - {D_i}^2)$$

The area of the pipe is

$$A = Ld_o$$

Equate.

$$L = \frac{\pi({D_o}^2 - {D_i}^2)}{4d_o} = 33 \text{ m}$$

f. Hollow pipe is used instead of solid conductors in order to allow for cooling of the magnet. If the resistivity of copper is $\rho = 1.7 \times 10^{-8}\Omega \cdot \text{m}$, determine the electrical resistance of one spiral.

Solution

$$r_s = \frac{\rho L}{A}$$

where A is the cross sectional area of the pipe, or

$$A = \frac{\pi}{4}({d_o}^2 - {d_i}^2) = 7.1 \times 10^{-6} \text{ m}^2$$

Combine with above, and

$$r_s = \frac{\rho}{d}\frac{{D_o}^2 - {D_i}^2}{{d_o}^2 - {d_i}^2} = 0.079\ \Omega$$

g. There are $N = 24$ coils stacked on top of each other. Tap water with an initial temperature of $T_c = 18°$ C enters the spiral through the copper pipe to keep it from over heating; the water exits at a temperature of $T_h = 31°$ C. The copper pipe carries a direct 45 Amp current in order to generate the necessary magnetic field. At what rate must the cooling water flow be provided to the electromagnet? Express your answer in liters per second with only one significant digit. The specific heat capacity of water is 4200 J/°C · kg; the density of water is 1000 kg/m^3.

Solution

The rate of heat generation in the coils is given by

$$P = I^2 R = I^2 N r_s = 3850 \text{ W}$$

This must be dissipated via the increase in water temperature,

$$P = C \Delta T Q$$

where C is the specific heat capacity in liters, and Q is the flow rate in liters per second. But since one liter of water is one kilogram, we can use either C.

Combine,

$$Q = \frac{I^2 N r}{C \Delta T} = 0.07 \text{ l/s}$$

h. The protons are fired at a target consisting of Fluorine atoms ($Z = 9$). What is the distance of closest approach to the center of the Fluorine nuclei for the protons? You can assume that the Fluorine does not move.

Solution

Conserve energy:

$$qV = \frac{1}{4\pi\epsilon_0} \frac{Zq^2}{r}$$

where r is closest approach.

Then

$$r = \frac{1}{4\pi\epsilon_0} \frac{Zq}{V} = 2.59 \times 10^{-14} \text{ m}$$

Since this is about the size of a Fluorine nucleus, we can potential get a nuclear reaction. Actually, the important reaction occurs at about 380 kV.

Exam Statistics

Question	A1	A2	A3	A4	B1	B2	Total
Mean	11	3	5	6	12	15	53
Standard Deviation	7	4	5	7	10	14	29
Maximum	25	21	25	25	42	50	163
Upper Quartile	18	4	9	8	18	26	73
Median	10	2	5	5	10	12	49
Lower Quartile	4	0	0	0	3	1	30
Minimum	0	0	0	0	0	0	5

Some Trivia:

- California had 121 test takers
- New Jersey had 42
- Texas had 38
- Florida, Illinois, Massachusetts, Maryland, New York, and Virginia each had between one dozen and two dozen test takers
- Alabama, Alaska, Colorado, Idaho, Iowa, Louisiana, Montana, North Dakota, South Dakota, and Wyoming did not have a test taker this year.