AAPT UNITED STATES PHYSICS TEAM
AIP 2024

2024 $F = ma$ Exam

25 QUESTIONS - 75 MINUTES

INSTRUCTIONS

DO NOT OPEN THIS TEST UNTIL YOU ARE TOLD TO BEGIN

- Use $g = 10$ N/kg throughout, unless otherwise specified.
- You may write in this question booklet and the scratch paper provided by the proctor.
- This test has 25 multiple choice questions. Select the best response to each question, and use a No.2 pencil to completely fill the box corresponding to your choice. If you change an answer, completely erase the previous mark. Only use the boxes numbered 1 through 25 on the answer sheet.
- All questions are equally weighted, but are not necessarily equally difficult.
- You will receive one point for each correct answer, and zero points for each incorrect or blank answer. There is no additional penalty for incorrect answers.
- You may use a hand-held calculator. Its memory must be cleared of data and programs. You may use only the basic functions found on a simple scientific calculator. Calculators may not be shared. Cell phones may not be used during the exam or while the exam papers are present. You may not use any external references, such as books or formula sheets.
- The question booklet, answer sheet and scratch paper will be collected at the end of this exam.
- **To maintain exam security, do not communicate any information about the questions or their solutions until after February 24, 2024.**

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We acknowledge the following people for their contributions to this year's exams (in alphabetical order):

Tengiz Bibilashvili, Orhun Ciftcioglu, Kellan Colburn, Natalie LeBaron, Brian Skinner, Elena Yudovina and Kevin Zhou

1. An archer fires an arrow from the ground so that it passes through two hoops, which are both a height h above the ground. The arrow passes through the first hoop one second after the arrow is launched, and through the second hoop another second later. What is the value of h?

(A) 5 m

$\boxed{\text{B}}$ 10 m

(C) 12 m

(D) 15 m

(E) There is not enough information to decide.

The times t_1 and t_2 satisfy the equation $gt^2/2 - v_0 t + h = 0$, where v_0 is the initial upward velocity. Therefore, $t^2 - 2v_0 t/g + 2h/g = (t - t_1)(t - t_2)$, from which we read off $t_1 t_2 = 2h/g$. Plugging in the numbers gives gives $h = 10\,\text{m}$.

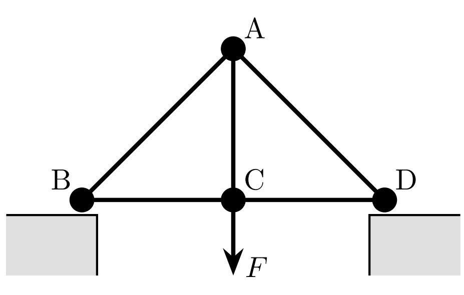
Alternatively, notice that if the arrow is at the same height at time $t_1 = 1\,\text{s}$ and $t_2 = 2\,\text{s}$, then it must have reached the top of its trajectory at time $t = 1.5\,\text{s}$, so it was launched with vertical velocity $15\,\text{m/s}$. In the first second, the average vertical velocity is therefore $(15 + 5)/2 = 10\,\text{m/s}$, giving $h = 10\,\text{m}$.

2. An amusement park ride consists of a circular, horizontal room. A rider leans against its frictionless outer walls, which are angled back at 30° with respect to the vertical, so that the rider's center of mass is 5.0 m from the center of the room. When the room begins to spin about its center, at what angular velocity will the rider's feet first lift off the floor?

$\boxed{\text{A}}$ 1.9 rad/s (B) 2.3 rad/s (C) 3.5 rad/s (D) 4.0 rad/s (E) 5.6 rad/s

When the rider's feet just lift off, the normal force from the floor vanishes, so the riders only experience a normal force from the walls. That normal force must both cancel the downward gravitational force mg and provide the inward centripetal force $mv^2/R = m\omega^2 R$. Thus, $\tan\theta = g/\omega^2 R$. Solving for ω using $\tan 30° = 1/\sqrt{3}$ gives $\omega = 1.86\,\text{rad/s}$.

3. A simple bridge is made of five thin rods rigidly connected at four vertices.



The ground is frictionless, so that it can only exert vertical normal forces at B and D. The weight of the bridge is negligible, but a person stands at its middle, exerting a downward force F at vertex C. In static equilibrium, each rod can be experiencing either tension or compression. Which of the following is true?

(A) Only the vertical rod is in tension.

(B) Only the horizontal rods are in tension.

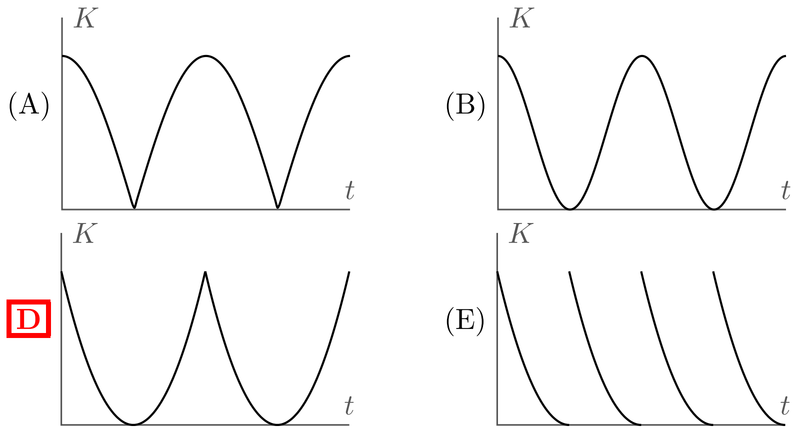
(C) Both the vertical rod and the diagonal rods are in tension.

$\boxed{\text{D}}$ Both the vertical rod and the horizontal rods are in tension.

(E) All of the rods are in tension.

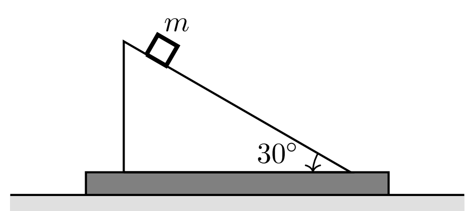
To balance vertical forces at C, the vertical rod has to be in tension. To balance vertical forces at A, the diagonal rods have to be in compression. And to balance horizontal forces at B and D, the horizontal rods have to be in tension.

4. A bouncy ball is thrown vertically upward from the ground. Air resistance is negligible, and the ball's collisions with the ground are perfectly elastic. Which of the following shows the kinetic energy of the ball as a function of time? Assume the collisions are too quick for their duration to be seen in the plot.



When the ball is in the air, its velocity is $v(t) = v_0 - gt$, so its kinetic energy is proportional to $v^2 = (v_0 - gt)^2$, which is a concave up parabola. When the ball bounces, its speed stays the same, so the kinetic energy stays the same and there's no discontinuity.

5. A massless inclined plane with angle 30° to the horizontal is fixed to a scale. A block of mass m is released from the top of the plane, which is frictionless.

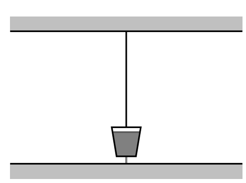


As the block slides down the plane, what is the reading on the scale?

(A) $\sqrt{3}\,mg/4$ (B) $mg/2$ $\boxed{\text{C}}$ $3mg/4$ (D) $\sqrt{3}\,mg/2$ (E) mg

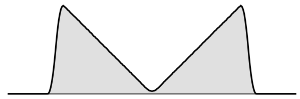
The scale reading is equal to the vertical component of the normal force of the block on the plane. The magnitude of the normal force is $mg\cos\theta$, and the vertical component is $\cos\theta$ times this, so the reading is $mg\cos^2(30°) = 3mg/4$.

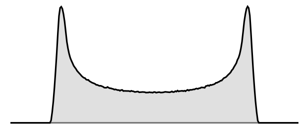
6. A pendulum is made with a string and a bucket full of water. When the string is vertical, the bottom of the bucket is near the ground.

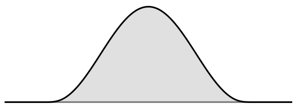


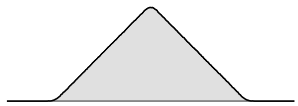
Then, the pendulum is set swinging with a small amplitude, and a very small hole is opened at the bottom of the bucket, which leaks water at a constant rate. After a few full swings, which of the following best shows the amount of water that has landed on the ground as a function of position?

(A) 

(B) 

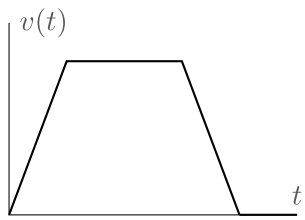
$\boxed{\text{C}}$ 

(D) 

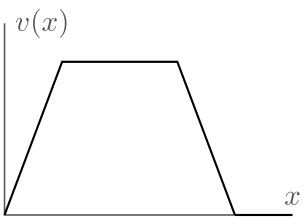
(E) 

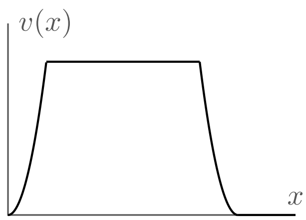
The amount of water at each location is proportional to how long the bucket spends there, so it is inversely proportional to the velocity. Therefore, it is the lowest (but nonzero) at the middle, where the bucket has maximum velocity, and highest at the endpoints, where the bucket is turning around.

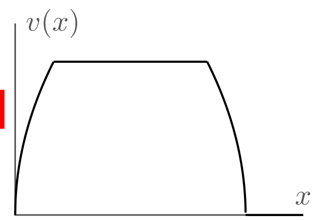
7. A particle travels in a straight line. Its velocity as a function of time is shown below.

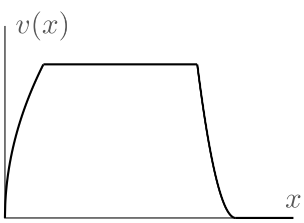


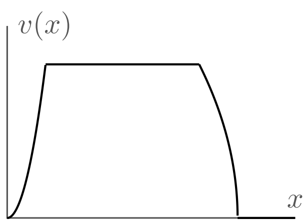
Which of the following shows the velocity as a function of distance x from its initial position?

(A) 

(B) 

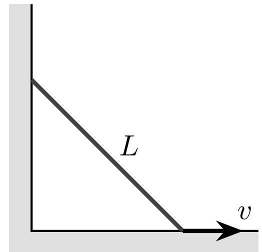
$\boxed{\text{C}}$ 

(D) 

(E) 

The particle accelerates uniformly, then travels at a constant velocity for some time, then decelerates uniformly down to resting. As a function of distance, then, the velocity must also increase, then stay constant for some time, then decrease down to 0. Since velocity vs. time is symmetric, velocity vs. distance must be symmetric as well; if the movie were played backwards, it should look the same. When the particle is accelerating uniformly, its velocity is proportional to t, while the distance is proportional to t^2 (e.g. as in free fall): consequently, velocity is proportional to the square root of distance.

8. A rod of length L is sliding down a frictionless wall.



When the rod makes an angle of 45° to the horizontal, the bottom of the rod has speed v. At this moment, what is the speed of the middle of the rod?

(A) $v/2$ $\boxed{\text{B}}$ $v/\sqrt{2}$ (C) v (D) $\sqrt{2}\,v$ (E) $2v$

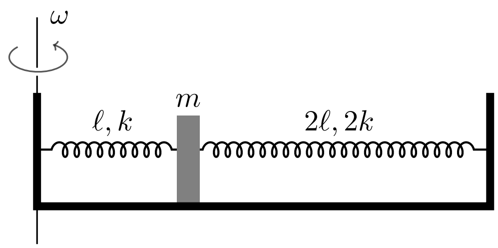
The bottom of the rod has rightward velocity v, so by symmetry, the top of the rod has downward velocity v. The velocity of the middle of the rod is the average of these velocities, and thus has magnitude $v/\sqrt{2}$.

9. When a car's brakes are fully engaged, it takes 100 m to stop on a dry road, which has coefficient of kinetic friction $\mu_k = 0.8$ with the tires. Now suppose only the first 50 m of the road is dry, and the rest is covered with ice, with $\mu_k = 0.2$. What total distance does the car need to stop?

(A) 150 m (B) 200 m $\boxed{\text{C}}$ 250 m (D) 400 m (E) 850 m

When the road is dry, the work done on the car by friction is equal to $\mu_d mgx$, where μ_d is the coefficient of friction with the dry road and $x = 100$ m is the braking distance. This work should be equal to the car's initial kinetic energy, $mv^2/2$. Equating them gives $v^2/2 = \mu_d gx$. When the car partially skids on ice, this same energy balance is $mv^2/2 = \mu_d gx/2 + \mu_i gy$, where μ_i is the coefficient of friction on ice and y is the portion of the braking distance that takes place on ice. Inserting the result for $v^2/2$ from the previous equation and solving for y gives $y = (x/2) \times (\mu_d/\mu_i) = 200$ m. So the total braking distance is $50 + 200 = 250$ m.

10. A block of mass m is connected to the walls of a frictionless box by two massless springs with relaxed lengths ℓ and 2ℓ, and spring constants k and $2k$ respectively. The length of the box is 3ℓ. The system rotates with a constant angular velocity ω about one of its walls.



Suppose the block stays at a constant distance r from the axis of rotation, without touching either of the walls. What is the value of r?

(A) $\dfrac{2k\ell}{2k - m\omega^2}$ (B) $\dfrac{2k\ell}{2k + m\omega^2}$ (C) $\dfrac{2k\ell}{3k + m\omega^2}$ $\boxed{\text{D}}$ $\dfrac{3k\ell}{3k - m\omega^2}$ (E) $\dfrac{3k\ell}{3k + m\omega^2}$

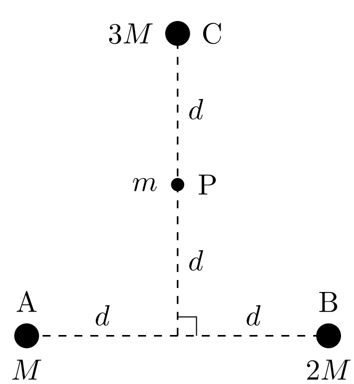
We balance forces on the block in the reference frame rotating with the box, to give $m\omega^2 r = k(r - \ell) + 2k(r - \ell)$. Solving for r gives the answer.

11. Two hemispherical shells can be pressed together to form a airtight sphere of radius 40 cm. Suppose the shells are pressed together at a high altitude, where the air pressure is half its value at sea level. The sphere is then returned to sea level, where the air pressure is 10^5 Pa. What force F, applied directly outward to each hemisphere, is required to pull them apart?

$\boxed{\text{A}}$ 25,000 N (B) 50,000 N (C) 100,000 N (D) 200,000 N (E) 400,000 N

The inward pressure force on each hemisphere is $\pi r^2 \Delta P$, where $r = 40\,\text{cm}$ and $\Delta P = (10^5\,\text{Pa})/2$. Plugging in the numbers gives a force of $25,000\,\text{N}$.

12. A space probe with mass m at point P traverses through a cluster of three asteroids, at points A, B, and C. The masses and locations of the asteroids are shown below.



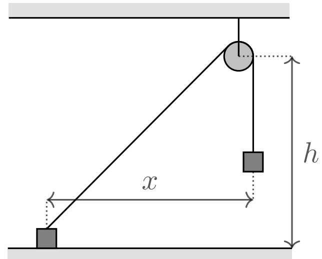
What is the torque on the probe about point C?

$\boxed{\text{A}}$ $\frac{1}{2\sqrt{2}}\frac{GMm}{d}$ (B) $\frac{1}{2}\frac{GMm}{d}$ (C) $\frac{1}{\sqrt{2}}\frac{GMm}{d}$ (D) $\frac{GMm}{d}$ (E) $\frac{\sqrt{2}\,GMm}{d}$

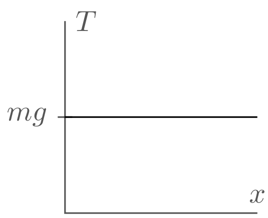
Only asteroids A and B contribute to the torque. Using the standard formula for torque, we have

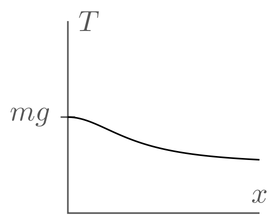
$$\tau = \frac{GMm}{(\sqrt{2}\,d)^2}\left(2\,\frac{d}{\sqrt{2}} - \frac{d}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\frac{GMm}{d}.$$

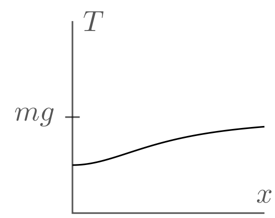
13. Two frictionless blocks of mass m are connected by a massless string which passes through a fixed massless pulley, which is a height h above the ground. Suppose the blocks are initially held with horizontal separation x, and the length of the string is chosen so that the right block hangs in the air as shown.

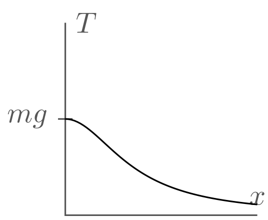


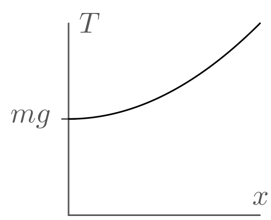
If the blocks are relased, the tension in the string immediately afterward will be T. Which of the following shows a plot of T versus x?

(A)


$\boxed{\text{B}}$


(C)


(D)


(E)


This problem can be solved with limiting cases. When $x = 0$, this reduces to an Atwood's machine with equal masses. Both masses are static, so the tension $T = mg$. When $x \to \infty$, the string becomes horizontal, so equating the accelerations of the two masses gives

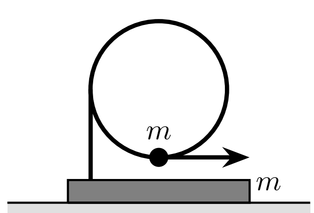
$$a = g - \frac{T}{m} = \frac{T}{m}$$

which yields $T = mg/2$. The only choice satisfying these limits is (B). Alternatively, with a little trigonometry, you can show that the tension in general is

$$T = \frac{h^2 + x^2}{h^2 + 2x^2}\, mg.$$

Note that in general, the tension also depends on the initial speeds of the blocks. In this problem, we have made the simplifying assumption that the initial speeds vanish.

14. A bead of mass m can slide frictionlessly on a vertical circular wire hoop of radius 20 cm.



The hoop is attached to a stand of mass m, which can slide frictionlessly on the ground. Initially, the bead is at the bottom of the hoop, the stand is at rest, and the bead has velocity $2\,\mathrm{m/s}$ to the right. At some point, the bead will stop moving with respect to the hoop. At that moment, through what angle along the hoop has the bead traveled?

(A) 30° (B) 45° $\boxed{\text{C}}$ 60° (D) 90° (E) 120°

When the bead stops moving with respect to the hoop, they must have the same velocity. By conservation of horizontal momentum, that velocity must be half the initial velocity. Thus, by conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}(2m)(v/2)^2 + mg\Delta h$$

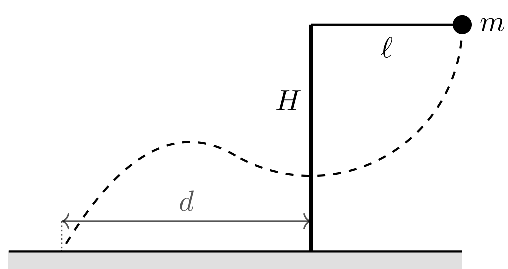
where Δh is the change in height of the bead. Solving gives $\Delta h = 10\,\mathrm{cm}$, which implies the bead slides by 60° along the hoop.

15. The viscous force between two plates of area A, with relative speed v and separation d, is $F = \eta A v/d$, where η is the viscosity. In fluid mechanics, the Ohnesorge number is a dimensionless number proportional to η which characterizes the importance of viscous forces, in a drop of fluid of density ρ, surface tension γ, and length scale ℓ. Which of the following could be the definition of the Ohnesorge number?

(A) $\dfrac{\eta\ell}{\sqrt{\rho\gamma}}$ (B) $\eta\ell\sqrt{\dfrac{\rho}{\gamma}}$ (C) $\eta\sqrt{\dfrac{\rho}{\gamma\ell}}$ (D) $\eta\sqrt{\dfrac{\rho\ell}{\gamma}}$ **E** $\dfrac{\eta}{\sqrt{\rho\gamma\ell}}$

From the definition of viscosity, the units of η are kg m/s^2. In addition, the units of ρ are kg/m^3 and the units of ℓ are m. As for the surface tension, it is a force per length, so its units are kg/s^2. We thus conclude that $\sqrt{\rho\gamma\ell}$ has the same units as η, so that choice (E) is the only possibility proportional to η that is properly dimensionless.

16. A child of mass m holds onto the end of a massless rope of length ℓ, which is attached to a pivot a height H above the ground. The child is released from rest when the rope is straight and horizontal.

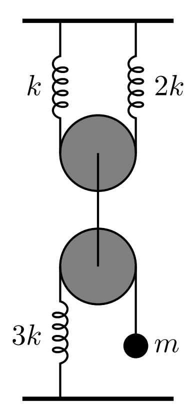


At some point, the child lets go of the rope, flies through the air, and lands on the ground a horizontal distance d from the pivot. On Earth, the maximum possible value of d is d_E. If the setup is moved to the Moon, which has 1/6 the gravitational acceleration, what is the new maximum possible value of d?

(A) $d_E/6$ (B) $d_E/\sqrt{6}$ **C** d_E (D) $\sqrt{6}\,d_E$ (E) $6\,d_E$

This is most easily solved by dimensional analysis: d has to be some function of l, H, m, and g. Of these, g is the only quantity with seconds in it; since d doesn't have any seconds in it, the distance cannot depend on the value of g.

17. Consider the following system of massless and frictionless pulleys, ropes, and springs.



Initially, a block of mass m is attached to the end of a rope, and the system is in equilibrium. Next the block is doubled in mass, and the system is allowed to come to equilibrium again. During the transition between these equilibria, how far does the end of the rope (where the block is suspended) move?

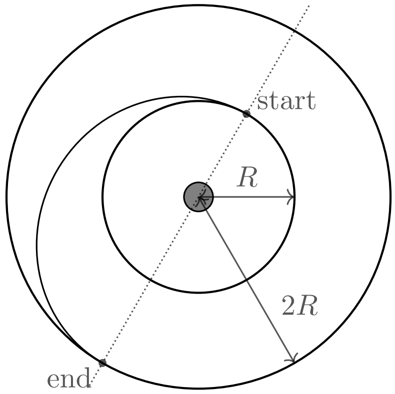
(A) $\frac{7}{12}\frac{mg}{k}$ (B) $\frac{11}{12}\frac{mg}{k}$ (C) $\frac{13}{12}\frac{mg}{k}$ (D) $\frac{7}{6}\frac{mg}{k}$ $\boxed{\text{E}}$ $\frac{11}{6}\frac{mg}{k}$

When the block is doubled in mass, it increases the tension in the bottom string by mg, so the third spring's length increases by $\Delta x_3 = mg/3k$. Since the bottom pulley has two ropes pulling down on it, the downward force on it increases by $2mg$, which means the upward force on the top pulley also has to increase by $2mg$, so the tension in the top string also increases by mg. Then the first and second springs increases in length by $\Delta x_1 = mg/k$ and $\Delta x_2 = mg/2k$.

Finally, by considering how changes in length of each spring affect the position of the mass, we conclude

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 = \frac{11mg}{6k}.$$

18. A satellite is initially in a circular orbit of radius R around a planet of mass M. It fires its rockets to instantaneously increase its speed by Δv, keeping the direction of its velocity the same, so that it enters an elliptical orbit whose maximum distance from the planet is $2R$.



What is the value of Δv? (Hint: when the satellite is in an elliptical orbit with semimajor axis a, its total energy per unit mass is $-GM/2a$.)

(A) $0.08\sqrt{\frac{GM}{R}}$ $\boxed{\text{B}}$ $0.15\sqrt{\frac{GM}{R}}$ (C) $0.22\sqrt{\frac{GM}{R}}$ (D) $0.29\sqrt{\frac{GM}{R}}$ (E) $0.41\sqrt{\frac{GM}{R}}$

The initial speed in the circular orbit is $v_i = \sqrt{GM/R}$. In the elliptical orbit, the semimajor axis is $a = 3R/2$, so the total energy per satellite mass is

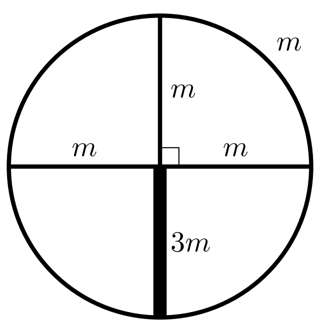
$$-\frac{GM}{3R} = -\frac{GM}{R} + \frac{1}{2}v_f^2$$

which implies

$$v_f^2 = \frac{4}{3}\frac{GM}{R}.$$

Computing $v_f - v_i$ gives the answer.

19. A wheel of radius R has a thin rim and four spokes, each of which have uniform density.



The entire rim has mass m, three of the spokes each have mass m, and the fourth spoke has mass $3m$. The wheel is suspended on a horizontal frictionless axle passing through its center. If the wheel is slightly rotated from its equilibrium position, what is the angular frequency of small oscillations?

$\boxed{\text{A}}$ $\sqrt{\frac{g}{3R}}$ (B) $\sqrt{\frac{g}{2R}}$ (C) $\sqrt{\frac{2g}{3R}}$ (D) $\sqrt{\frac{g}{R}}$ (E) $\sqrt{\frac{7g}{6R}}$

This system is a physical pendulum. The moment of inertia about the center is

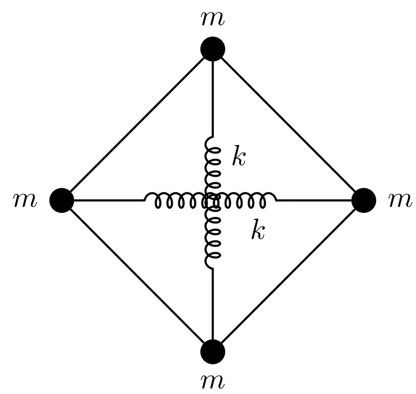
$$I = mR^2 + \frac{1}{3}\left(m + m + m + 3m\right)R^2 = 3mR^2.$$

When the system is rotated a small angle θ from equilibrium, the restoring torque comes from the difference in masses of the top and bottom spokes, so

$$\tau = -(3m - m)g\frac{R}{2}\sin\theta \approx -mgR\theta.$$

As a result, we have $\alpha = -(g/3R)\theta$ from which we read off $\omega = \sqrt{g/3R}$.

20. Four massless rigid rods are connected into a quadrilateral by four hinges. The hinges have mass m, and allow the rods to freely rotate. A spring of spring constant k is connected across each of the diagonals, so that the springs are at their relaxed length when the rods form a square.



Assume the springs do not interfere with each other. If the square is slightly compressed along one of its diagonals, its shape will oscillate over time. What is the period of these oscillations?

(A) $2\pi\sqrt{\frac{m}{4k}}$ $\boxed{\text{B}}$ $2\pi\sqrt{\frac{m}{2k}}$ (C) $2\pi\sqrt{\frac{m}{k}}$ (D) $2\pi\sqrt{\frac{2m}{k}}$ (E) $2\pi\sqrt{\frac{4m}{k}}$

Suppose that one of the masses has moved a distance x from its original position, and it currently has speed v. By symmetry, all of the other masses have also moved a distance x, and also have a speed v, so the total kinetic energy is $K = (4m)v^2/2$. Each spring is either stretched or compressed by a distance $2x$, so the total elastic potential energy is $U = (8k)x^2/2$. Thus, the kinetic and potential energy are the same as that of a system of mass $m_{\text{eff}} = 4m$ on a spring of spring constant $k_{\text{eff}} = 8k$, so the angular frequency is $\sqrt{k_{\text{eff}}/m_{\text{eff}}} = \sqrt{2k/m}$.

21. A syringe is filled with water of density ρ and negligible viscosity. Its body is a cylinder of cross-sectional area A_1, which gradually tapers into a needle with cross-sectional area $A_2 \ll A_1$. The syringe is held in place and its end is slowly pushed inward by a force F, so that it moves with constant speed v. Water shoots straight out of the needle's tip. What is the approximate value of F?

(A) $\rho v^2 A_1$ (B) $\frac{\rho v^2 A_1^2}{2A_2}$ (C) $\frac{\rho v^2 A_1^2}{A_2}$ $\boxed{\text{D}}$ $\frac{\rho v^2 A_1^3}{2A_2^2}$ (E) $\frac{\rho v^2 A_1^3}{A_2^2}$

If the water has speed v' when it leaves the needle tip, then $A_1 v = A_2 v'$. The water outside the syringe has atmospheric pressure, so by Bernoulli's principle, the pressure P of the water inside the syringe obeys

$$P - P_{\text{atm}} = \frac{1}{2}\rho(v'^2 - v^2) \approx \frac{1}{2}\rho v'^2 = \frac{\rho v^2 A_1^2}{2A_2^2}.$$

The required force is the area times the pressure difference.

Incidentally, if you don't know Bernoulli's principle, you can also solve the problem by equating the power Fv with the rate of change of kinetic energy of the water. However, if you try to equate the force F to the rate of change of momentum of the water, you'll get $F = \rho v^2 A_1^2/A_2$ which is incorrect. The reason this method is wrong is that it doesn't account for the other forces needed to hold the syringe in place.

22. A spherical shell is made from a thin sheet of material with a mass per area of σ. Consider two points, P_1 and P_2, which are close to each other, but just inside and outside the sphere, respectively. If the accelerations due to gravity at these points are $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, what is the value of $|\mathbf{g}_1 - \mathbf{g}_2|$?

(A) $\pi G\sigma$ (B) $4\pi G\sigma/3$ (C) $2\pi G\sigma$ $\boxed{\text{D}}$ $4\pi G\sigma$ (E) $8\pi G\sigma$

Suppose the shell has radius R. By the shell theorem, the gravitational field vanishes inside, so $\mathbf{g}_2 = 0$. The gravitational field just outside has magnitude $g_1 = GM/R^2$ where $M = 4\pi\sigma R^2$ is the total mass of the shell. Thus, the unknown variable R cancels out, and the answer is $4\pi G\sigma$.

Note that this is actually a universal statement. Above, we found that it didn't depend on the sphere's radius, but more generally, it is true regardless of the shell's shape. The reason is that $|\mathbf{g}_1 - \mathbf{g}_2|$ is solely determined by the mass near P_1 and P_2, and if one zooms in enough to a patch of any surface, then it always looks like an infinite plane, so the answer doesn't depend on the shape of the surface away from that patch.

23. Collisions between ping pong balls and paddles are not perfectly elastic. Suppose that if a player holds a paddle still and drops a ball on top of it from any height h, it will bounce back up to height $h/2$. To keep the ball bouncing steadily, the player moves the paddle up and down, so that it is moving upward with speed 1.0 m/s whenever the ball hits it. What is the height to which the ball is bouncing?

(A) 0.21 m
(B) 0.45 m
(C) 1.0 m
$\boxed{\text{D}}$ 1.7 m
(E) There is not enough information to determine the height.

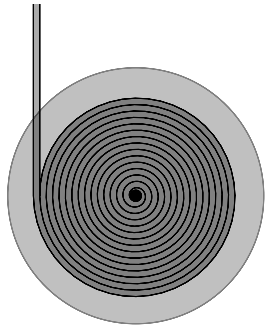
In the reference frame of the paddle, the ball loses half its energy during the collision, so the speed of the ball immediately after the collision is $1/\sqrt{2}$ of its speed immediately before the collision. Let v be the speed of the ball relative to the ground immediately before the collision, then we have $(v - 1\,\text{m/s}) = (v + 1\,\text{m/s})/\sqrt{2}$, or $v = 5.83\,\text{m/s}$. The height of the bouncing is $v^2/(2g) = 1.7\,\text{m}$.

24. When a projectile falls through a fluid, it experiences a drag force proportional to the product of its cross-sectional area, the fluid density ρ_f, and the square of its speed. Suppose a sphere of density $\rho_s \gg \rho_f$ of radius R is dropped in the fluid from rest. When the projectile has reached half of its terminal velocity, which of the following is its displacement proportional to?

(A) $R\sqrt{\rho_s/\rho_f}$ $\boxed{\text{B}}$ $R\,\rho_s/\rho_f$ (C) $R\,(\rho_s/\rho_f)^{3/2}$ (D) $R\,(\rho_s/\rho_f)^2$ (E) $R\,(\rho_s/\rho_f)^3$

If the ball falls a distance d, then its speed in the absence of drag would be $v \sim \sqrt{gd}$. Thus, if the drag force has not had a large effect yet, then its magnitude would be $F_d \sim \rho_f A v^2 \sim \rho_f R^2 g d$, while the gravitational force has magnitude $F_g \sim \rho_s R^3 g$. The object begins to approach terminal velocity when these are comparable, $\rho_f R^2 g d \sim \rho_s R^3 g$, which implies $d \sim (\rho_s/\rho_f)R$.

25. A yo-yo consists of two massive uniform disks of radius R connected by a thin axle. A thick string is wrapped many times around the axle, so that the end of the string is initially a distance R from the axle. Then, the end of the string is held in place and the yo-yo is dropped from rest. Assume that energy losses are negligible, and that the string has negligible mass and always remains vertical. Below, we show a cross-section of the yo-yo partway through its descent.



Between the moment the yo-yo is released and the moment the string completely unwinds, which of the following is true regarding the yo-yo's acceleration?

(A) It is always zero.

(B) It points downward, but decreases in magnitude over time.

(C) It points downward and has constant magnitude.

(D) It points downward, but increases in magnitude over time.

$\boxed{\text{E}}$ None of the above.

As the string unwinds, the radius r of the string remaining about the axle decreases. Specifically, if the yo-yo has fallen a distance z, and the string has total length ℓ, then

$$\frac{r^2}{R^2} = \frac{\ell - z}{\ell}.$$

This change in r affects how the energy of the yo-yo is divided between translational and rotational motion. If the discs have total mass m, conservation of energy gives

$$mgz = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where $I = mR^2/2$ but $\omega = v/r$. Solving for v gives

$$v = \sqrt{\frac{2gz}{1 + \frac{\ell}{2(\ell - z)}}}$$

which has a maximum at around $z = 0.6\ell$. Intuitively, in the beginning the yo-yo has roughly constant acceleration downward, but as the string unwinds, more of the energy goes into the yo-yo's rotational motion. In fact, the translational speed of the yo-yo goes to zero when the string completely unwinds, at which point it's just spinning in place! Therefore, during the latter part of its fall, the yo-yo has *upward* acceleration, so the answer is "none of the above".

This remains true regardless of the details of the yo-yo, because it only depends on the fact that the radius r starts equal to R and then becomes very small, since the axle is thin. In particular, changing how r depends on z, or the moment of inertia of the disc, won't change the answer.